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RESEARCH ARTICLE

Deployment in wireless sensor networks by parallel and cooperative parallel artificial bee colony algorithms

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ABSTRACT

Increasing number of cores in a processor chip and decreasing cost of distributed memory based system setup have led to emerge of a new work theme in which the main concern focused on the parallelization of the well-known algorithmic approaches for utilizing the computational power of the current architectures. In this study, the performances of the conventional parallel and cooperative model based parallel Artificial Bee Colony (ABC) algorithms on the deployment problem related to the wireless sensor networks were investigated. The results obtained from the experimental studies showed that parallelized ABC algorithm with the cooperative model is capable of finding similar or better coverage ratios with the increased convergence speeds than its serial counterpart and parallelized implementation in which the emigrant is chosen as the best food source in the current subcolony.

1. Introduction

Wireless sensor networks including hundreds or sometimes thousands of stationary or mobile nodes have been used various times for industrial or military projects [1, 2]. Each of the sensor nodes is capable of sending or receiving data packages and gathering information from the environment or objects being tracked. [1, 2]. However, sensor nodes have limited computing abilities and storage spaces, their detection ranges are restricted with properties of the sensing units and finally required power for sensing and communication is maintained by a small battery which can not be recharged or changed easily.

By considering all of these limitations and budget constraints, the configuration and settlement of a wireless sensor network should be made in order to maximize the life or utilization time of the network and the area of interest [1, 2]. The life time and successfully covered area of a wireless sensor network are directly related to the positions of the sensor nodes. If all the sensor nodes are deployed to the monitoring area in a straightforward manner that concerns the highest coverage ratio, the requirements for changing the positions of the mobile nodes by consuming extra energy from the internal battery decrease and the overall network life-time is substantially extended [1, 2]. With the increased understanding about the relationship between the positions of the sensors and efficiency of the network, studies on the deployment of sensor nodes have attracted the researchers and different approaches for solving the sensor deployment problem have been proposed.

When the studies about the sensor deployment problem are investigated, it is clearly seen that evolutionary computing techniques are commonly used. Bhondekar et al. used Genetic algorithm (GA) as a placement methodology of sensor nodes with different operating modes [3]. They tried to optimize a fitness function in which operational energy, number of unconnected sensors, number of overlapping cluster-in-charge, field coverage and number of sensors per cluster-in-charge are used as constraints. While the operational energy, number of unconnected sensors, number of overlapping cluster-in-charge constraints should be minimized, field coverage and number of sensors per cluster-in-charge constraints should be maximized [3]. Okay and Ozdemir analyzed the performances of the Multi-objective Evolutionary Algorithm based on Decomposition (MOEA/D) and Fast and Elitist Genetic Algorithm (NSGA-II) on optimization of sensing coverage area and total travel distances of the mobile nodes [4]. Obtained results from the experimental studies with 25 mobile sensors tracking 50 targets distributed to a $100m \times 100m$ area showed that NSGA-II is produced more robust deployments compared to MOEA/D in terms of tracked objects [4].

Li and Lei proposed a sensor deployment technique based on the Particle Swarm Optimization (PSO) algorithm called IPSO [5]. Distribution of 40 mobile sensors to a $80m \times 80m$ grid area with IPSO algorithm significantly improved the coverage ratio calculated with the probabilistic detection model compared to the Virtual Force (VF) algorithm [5]. One of the first studies about the using ABC algorithm as a sensor deployment technique has been carried out by Ugdata et al [6]. Ugdata et al. modeled sensor deployment problem as a data clustering problem and number of sensor nodes was used on behalf of clusters and locations of the sensor nodes were matched with the centroids of clusters [6]. Ozturk et al. investigated solving capabilities of the ABC algorithm for dynamic deployment problem of wireless networks with the two different studies [7, 8]. In the first study of them, ABC algorithm was used in order to maximize the coverage ratio of the network containing 100 mobile sensors [7, 8]. In the second study, they compared ABC algorithm with the PSO algorithm on solving a dynamic deployment scenario in which 20 mobile sensors are tried to be positioned at the suitable locations within a $10,000m^2$ square region [7, 8]. Results from the experimental studies showed that ABC algorithm is capable of producing more qualified solutions than the PSO algorithm. Yu et al. solved deployment problem by utilizing a modified ABC algorithm named as FNF-BL-ABC [9]. In the FNF-BL-ABC algorithm, the original equation of the ABC algorithm used to generate candidate solutions for onlooker bee phase was changed with the forgetting (F) and neighbor (N) factors [9]. In addition to these, they introduced a probabilistic model called back propagation learning (BL) for determining whether a solution is abandoned or not in the scout bee phase. Simulation results in an ideal area and an area with obstacles showed that TNF-BL-ABC algorithm produces better coverage ratios than standard ABC algorithm and increases the convergence speed [9]. Yadav et al. changed the search equation used by the employed and onlooker bee phases of the standard ABC algorithm and tested the proposed ABC algorithm variant for dynamic positioning of sensor networks [10].

In this study, the performances of the parallelized ABC algorithms powered with the conventional and cooperative emigrant creation strategies for solving the deployment problem of sensor networks were analyzed. The improving effects of the cooperative emigrant creation strategy already seen in numerical optimization problems were also investigated through sensor deployment problem. The rest of the paper is organized as follows: In the second section, definition of the sensor deployment problem, coverage calculation and sensor detection approach called binary detection are given. Fundamental steps of the ABC algorithm and its parallelization according to the mentioned emigrant creation strategies are summarized in third and fourth sections, respectively. Experimental studies with different control parameters are presented in fifth section. Finally, conclusions and future works are given in the sixth section.

2. Deployment problem in wireless sensor networks

When a wireless sensor network is established, the main purposes of the settlement are to maximize the utilization period of the network and the area where the sensors successfully in communication with each other by sending information obtained from the tracked objects or environmental variables [5–9]. To maximize these two conflicting objectives, exact positions of the mobile and stationary sensor nodes should be determined carefully. However, there is usually no priori information about the area of interest or the targets being tracked [5–9].

By considering all of these limitations, sensor deployment can be defined as a problem for which the coverage of the network is maximized by correctly positioning sensor nodes. When the sensor nodes are deployed, the coverage ratio of the network that shows the percentage of the successfully covered area is calculated as in the Eq. (1). In the Eq. (1), c_i is the coverage of the ith sensor in the set of sensors S and A is the size of the area [5–9].

$$CR = \frac{\bigcup c_i}{A}, i \epsilon S \tag{1}$$

If the area of interest is divided into equally sized subareas or grids, and P is a point corresponds to the corner of a grid at position (x, y), the Euclidean distance between the point P and the sensor s_i positioned at (x_i, y_i) is used to decide whether point P is in detection range of sensor s_i or not [5–9]. By taking the detection range of the sensor s_i as r and the Euclidean distance between the P and s_i as $d(P, s_i)$, the coverage of point P by s_i, $c_p(s_i)$, is equal to 1 if $d(P, s_i)$ is less than r, otherwise $c_p(s_i)$ is equal to 0. The binary sensor detection model used in the coverage calculation is given in the Eq. (2) for P and s_i [5–9].

$$c_i = \begin{Bmatrix} 1, & d(P, c_i) < r \\ 0, & d(P, c_i) \geq r \end{Bmatrix} \tag{2}$$

3. ABC algorithm and its adaptation to sensor deployment problem

In a real honey bee colony, an intelligent foraging behaviour is carried out by three groups of bees called employed, onlooker and scout bees, respectively [11–13]. Employed bees are responsible for finding new food sources around the previously visited ones and carry nectar to the hive. When an employed bee turns back to the hive, she shares the information about the nectar quality of the memorized food source, location and distance to the hive with the onlooker bees [11–13]. Onlooker bees wait on the hive and select food sources introduced by the employed bees. However, selection of a food source by an onlooker is actually not a random operation. If a food source introduced by an employed is rich in terms of nectar, it is highly possible that it attracts more onlooker bees compared with the poor sources [11–13]. After an onlooker bee selects a food source, she becomes an employed and continues the foraging operation as an employed. The final group of bees consists of scout bees and scout bees randomly search the environment to find an undiscovered food source.

By considering intelligent job division and foraging behaviours of bee colonies, Karaboga proposed a new population based optimization algorithm called ABC algorithm [11–13]. In ABC algorithm, positions of the food sources correspond to the possible solutions of the interested problem and the nectar quality of a food source is directly related to the appropriateness of the solution. ABC algorithm starts its optimization operations by randomly generating a set of food sources [14–16]. Assume that there are *SN* different food sources each of them contains D parameters, the *jth* parameter of the *ith* food source, shortly x_{ij}, can be generated between lower bound x_j^{min} and upper bound x_j^{max} as described in Eq. (3) [14–16].

$$x_{ij} = x_j^{min} + rand(0,1)(x_j^{max} - x_j^{min}) \tag{3}$$

When solving sensor deployment problem, a food source is matched with the positions of the sensors belonging to the created network and a food source or solution containing S wireless sensors can be represented by a specialized D dimensional vector in which each element is filled with location information of the sensor. In Fig. 1, a food source is illustrated for deployment of D wireless sensors into a two dimensional area.

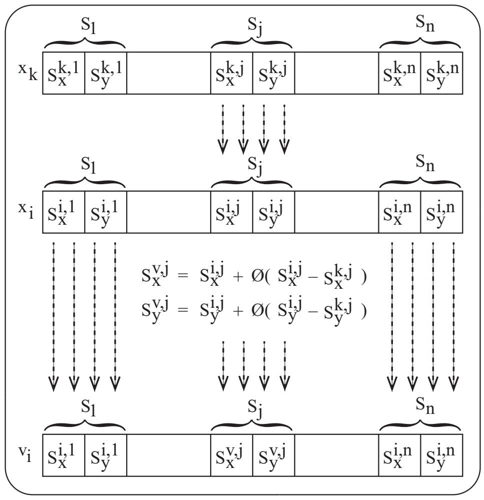


Figure 1. Representation of a solution for sensor deployment problem.

After generating initial food sources, each food source is associated only one employed bee. An employed bee is responsible with producing a candidate solution in the vicinity of the memorized food source by utilizing the Eq. (4) [17–19].

$$v_{ij} = x_{ij} + \phi_{ij}(x_{ij} - x_{kj}) \tag{4}$$

In Eq. (4), v_{ij} is the *jth* parameter of the candidate food source v_i. It should be noted that v_i is same with the x_i food source except the *jth* parameter. x_{ij} and x_{kj} are the *jth* parameters of the x_i and x_k solutions, respectively [19–23]. Finally, θ is a random coefficient between -1 and 1. If the $fit(v_i)$ fitness value of the v_i solution calculated by using the $obj(v_i)$ objective function value for a maximization problem as in the Eq. (5) is higher than the $fit(x_i)$ fitness value of the x_i food source, x_i food source is replaced with the v_i food source and the trial counter $trial_i$ showing how many times the x_i food source is not improved is set to zero. Otherwise, the same counter is incremented by one and its value is used to make a decision whether that food source is consumed or not [19–23].

$$fit(x_i) = \begin{Bmatrix} 1 + |obj(x_i)|; & obj(x_i) > 0 \\ 1/(1 + obj(x_i)); & obj(x_i) \leqslant 0 \end{Bmatrix} \quad (5)$$

When all the employed bees complete their operations and turn back to the hive, they share the information about the memorized food sources with the onlooker bees as mentioned before. Onlooker bees waiting on the hive select food sources and become employed foragers. However, each solution introduced by employed bees does not have equal chance for selection and qualified sources attract more onlookers. The relationship between choosability of a food source and its quality is modeled in ABC algorithm by assigning selection probability for each food source as calculated in Eq. (6) [19–23]. In Eq. (6), $p(x_i)$ shows the selection probability of the x_i solution with $fit(x_i)$ fitness value and it is clearly seen that $p(x_i)$ increases with the higher values of $fit(x_i)$. After a food source is chosen by an onlooker bee, this onlooker becomes an employed and produce a candidate solution using Eq. (4) [19–23].

$$p(x_i) = \frac{fit(x_i)}{\sum_j^{SN} fit(x_j)} \quad (6)$$

If a food source is not improved within employed and onlooker bee phases, a decision whether this food source is still consumed in the next cycle or not should be made to maintain the diversity of the solution set. In ABC algorithm, this decision is made by comparing the trial counters of the food sources with a control parameter called *limit*. A food source for which its trial counter exceeds the value of the *limit* parameter at most is abandoned and a scout bee is sent from the hive to discover a new food source as in the Eq. (3). In order to adjust exploration and exploitation characteristics of the algorithm, value of the *limit* parameter should be chosen carefully. For determining appropriate *limit* parameter of *SN* food sources when solving a D dimensional optimization problem, the formulation in Eq. (7) can be used [19–23].

$$\lceil a \times SN \times D \rceil \; and \; a \in \mathbb{Q}^+ \quad (7)$$

By considering the properties of the employed, onlooker and scout bee phases, the fundamental steps of the ABC algorithm and cyclical relationship between the mentioned bee phases are summarized in the Fig. 2.

```
Initialization:
  Assign values to limit and MFE parameters.
  Generate SN initial food source by using Eq. (3).
  Set evalCounter to zero.
Repeat
//Employed bee phase
   for i ← 1 ... SN do
     if evalCounter < MFE then
       Generate new solution x_new by using Eq. (4).
       Calculate fitness value of new solution.
       if fit(x_new) > fit(x_i) then
         Change x_i with x_new
       end if
       evalCounter ← evalCounter + 1
     end if
   end for
//Employed bee phase
//Onlooker bee phase
  sentBees ← 0, current ← 1
  Find probability values for each source by using Eq. (6).
  while sentBees ≠ SN and evalCounter < MFE do
    if p_current > rand(0,1) then
      sentBees ← sendBees + 1
      Generate new solution x_new by using Eq. (4).
      Calculate fitness value of new solution.
      if fit(x_new) > fit(x_i) then
        Change x_i with x_new
      end if
      evalCounter ← evalCounter + 1
    end if
    current ← ( current + 1 ) mod SN
  end while
//Onlooker bee phase
//Scout bee phase
  if evalCounter < MFE then
    Determine the abandoned food source using limit value.
    Generate a new source for the abandoned one by using Eq. (3).
    evalCounter ← evalCounter + 1
  end if
Until MFE is reached.
//Scout bee phase
```

Figure 2. Fundamental steps of the ABC algorithm.

4. Parallelization of ABC algorithm with conventional and cooperative model

Population based optimization algorithms including ABC algorithm are generally suitable for parallelization on distributed or shared memory based architectures. However, some steps of the algorithms require sequential operations and a limited set of modifications on the fundamental workflow of them should be made when they are tried to be parallelized. Dividing the whole colony into equally sized small colonies and evaluating them simultaneously on the different computing units are probably the most preferred parallelization approach [24–26]. However, number of bees in computing units is usually not enough compared to the serial implementations on single computing unit and parallelization does not go beyond a method that only focusing improvement on the execution times [24–26].

In order to address the problem to do with the number of bees in computing units, some solutions or individuals are migrated between subcolonies. The best solutions in each subcolony are the appropriate emigrant candidates and they usually change with the worst solutions in the neighbor subcolonies. This type of migration schema is the common part of the studies devoted to the parallelization and can be thought as the conventional approach [24–26]. However, if the best solutions can not be improved between subsequent migration periods, two or more copies of the same emigrant can be seen in the neighbor subcolony.

For increasing the efficiency of the emigrant solution and ensuring that the different emigrants are sent, an emigrant solution should be powered with other solution or solutions before it is sent to the neighbor subcolony. The mentioned idea about powering the best solution in a subcolony before migration is the main motivation of the cooperative model. In cooperative model, the best food source in a subcolony is strengthened by the more convenient parameters of the randomly chosen food source in the same subcolony. The Fig. 3 below illustrates the fundamental steps of the cooperative model in which neighborhood between subcolonies is determined by the ring topology.

```
Initialization:
  Assign values to limit and MFE parameters.
  Generate SN initial food source by using Eq. (1).
  Set evalCounter to zero and define a migPeriod.
  Determine numOfSubCol.
Repeat
After completion of a phase-triple
  if migPeriod is reached then
    if evalCounter < MFE then
      subColony ← index of current subcolony.
      x_random ← a random source in the (subColony)th subcolony.
      x_best, x_coop ← the best source in the (subColony)th subcolony.
      for i ← 1 ... D do
        Change x_coop,i with x_random,i
        Calculate the fitness value of x_coop
        evalCounter ← evalCounter + 1
        if fit(x_coop) < fit(x_best) then
          Change x_coop,i with x_best,i
        end if
      end for
  Send x_coop to the ((subColony + 1) mod numOfSubCol)th subcolony.
Until MFE is reached
```

Figure 3. Fundamental steps of the parallel ABC algorithm with cooperative model.

As seen from the fundamental steps of the parallel ABC algorithm with cooperative model, the best food source chosen as an emigrant for the current migration period is modified with the parameters of the randomly selected food source. If the *ith* parameter of the randomly selected food source increases the fitness value of the best food source, the *ith* parameter of the best food source is replaced with the corresponding parameter of the randomly selected food source. By utilizing this type of emigrant creation schema, the probability of sending qualified food sources as emigrants and the chance for consumption more qualified solutions are significantly increased.

However, it should be noted that generation of cooperative emigrant requires D times more fitness evaluations compared to the conventional emigrant creation schema. If the migration period and neighborhood topology are chosen by considering the computational burden of the cooperative emigrant creation approach, the speedup and efficiency values of the parallelized ABC algorithm with cooperative model get closer to the speedup and efficiency values of the parallelized ABC algorithm in which the emigrant is determined as the local best food source in the subcolony and then it is sent to the neighbor subcolony without modification.

5. Experimental studies

In order to analyze the performance of the conventional and cooperative emigrant creation schema for solving the sensor deployment problem, a set of experimental studies has bee carried out with 100 mobile sensors that should be positioned at the suitable locations on a $100m \times 100m$ area by considering the maximization of the coverage. For serial ABC algorithm, sABC algorithm, parallel ABC algorithm with the conventional emigrant creation strategy, pABC algorithm, and parallel ABC algorithm with the cooperative emigrant creation strategy, coop-pABC algorithm, the colony size was set to 20 and the *limit* parameter was chosen as 100 for the experiments [7, 8].

Neighborhood topology of the pABC and coop-pABC algorithms was ring and for each subcolony only one emigrant was generated. When an emigrant was sent to its neighbor subcolony, it was replaced with the worst solution found in the neighbor subcolony. The migration period (migration rate) that determines the frequency of the communication between subcolonies was set to 20 which means that after completion of a 20 employed-onlooker-scout bee phase triple, subcolonies exchange their emigrants according to the used neighborhood topology. sABC algorithm, pABC and coop-pABC algorithms with four subcolonies were tested independently until the maximum evaluation number reached to $1,000$, $2,000$ and $10,000$ on a system equipped with Intel i5 750 processor and 4 GB of RAM. sABC, pABC and coop-pABC algorithms were coded in C programming language and the required synchronization between subcolonies or processor cores were maintained by using the built-in function in pthreads library. Each of the algorithm was run 20 different times with random seeds and the means best coverage ratios and standard deviations related to the 20 runs were recorded and given in the Tables 1-3.

When the results given in the Tables 1-3 are investigated it is clearly seen that the the coop-pABC algorithm is capable of producing better mean coverage ratios compared to the pABC algorithm for all of the three experimental cases and the sABC algorithm for the two of three different experimental cases. By starting distribution of the cooperative emigrants, parallelized ABC algorithm improves the qualities of the solutions in each subcolony. Even though the differences between mean best coverage ratios of the algorithms are relatively small, the complex structure of the deployment problem and the difficulty on improving coverage value after determining positions of the some sensors should be remembered.

Table 1. Coverage values obtained by the sABC and pABC.

Evaluations	sABC		pABC	
	Mean	Std.Dev.	Mean	Std.Dev.
1,000	**0.88257**	**0.00410**	0.87507	0.00594
2,000	**0.91207**	0.00638	0.90887	**0.00483**
10,000	0.96755	**0.00226**	**0.96904**	0.00372

Table 2. Coverage values obtained by the sABC and coop-pABC.

Evaluations	sABC		coop-pABC	
	Mean	Std.Dev.	Mean	Std.Dev
1,000	**0.88257**	**0.00410**	0.87970	0.00457
2,000	0.91207	0.00638	**0.91530**	**0.00362**
10,000	0.96755	**0.00226**	**0.97063**	0.00553

Table 3. Coverage values obtained by the pABC and coop-pABC.

Evaluations	pABC		coop-pABC	
	Mean	Std.Dev.	Mean	Std.Dev
1,000	0.87507	0.00594	**0.87970**	**0.00457**
2,000	0.90887	0.00483	**0.91530**	**0.00362**
10,000	0.96904	**0.00372**	**0.97063**	0.00553

One of the main purposed with the parallelization of an algorithm is actually decreasing the execution times compared to the its serial implementation while protecting the qualities of the final solutions or results. For measuring the gain in the execution times, two important metrics called speedup and efficiency are commonly used. Speedup measure can be explained as a ratio between average execution times between serial and parallel implementations of the same algorithm and its maximum value can be equal to the number of cores or computing nodes of the cluster. If the speedup value of the parallelization is equal to the number of core or computing nodes, it is said that parallelization is linear. Efficiency metric is defined as a ratio between speedup and number of computing units used in the parallelization schema.

If the parallelization overhead stemmed from the mechanism such as synchronization, mutual exclusion can not be neglected, the maximum value of the efficiency can be relatively close to one. In Tables 4-7, average execution times of the sABC, pABC and coop-pABC algorithms, speedup and efficiency values for parallel implementations are given. As seen from the results given in Tables 4-7, conventional emigrant creation strategy reaches more desired speedup and efficiency values when compared to the cooperative emigrant creation strategy based parallelization approach. If the reduction in execution time is the main concern of the parallelization, the migration period should be carefully chosen to balance the qualities of the final solutions and speedup-efficiency values.

Table 4. Average execution times for sABC and pABC.

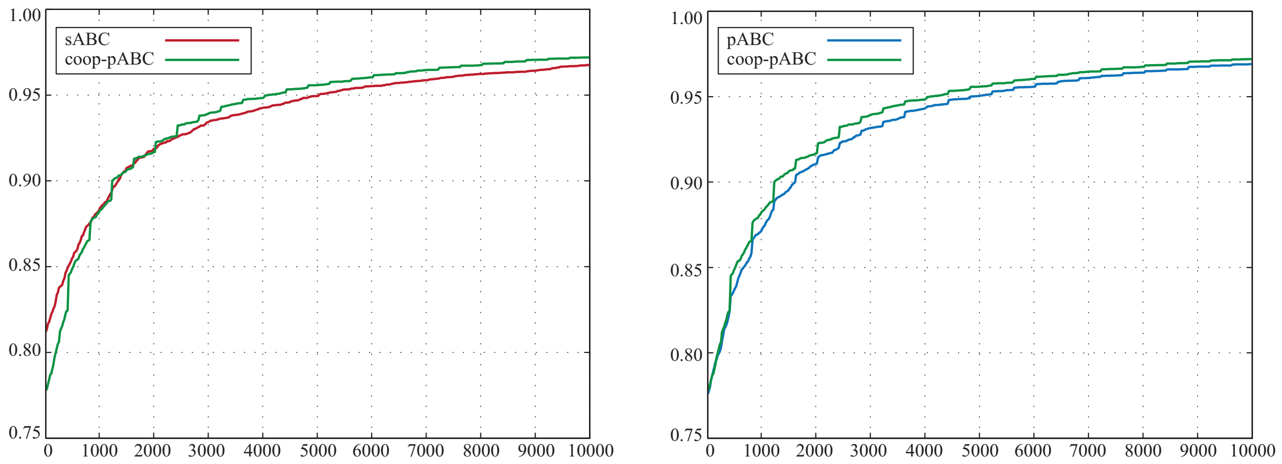
Evaluations	sABC		pABC	
	Mean	Std.Dev.	Mean	Std.Dev.
1,000	48.63031	2.01138	13.39241	0.60145
2,000	94.37129	3.73157	27.23307	1.18997
10,000	437.05957	11.35073	124.54752	3.29203

Table 5. Speedup and efficiency values of pABC.

Evaluations	ABC and pABC Algorithms	
	Speedup	Efficiency
1,000	3.63118	0.90779
2,000	3.46531	0.86633
10,000	3.50917	0.87729

Table 6. Average execution times for sABC and coop-pABC.

Evaluations	sABC		coop-pABC	
	Mean	Std.Dev.	Mean	Std.Dev.
1,000	48.63031	2.01138	18.78291	0.69383
2,000	94.37129	3.73157	37.47984	1.55957
10,000	437.05957	11.35073	187.71349	3.48997



(a) **(b)**

Figure 4. Convergence curves of sABC and coop-pABC (a) and pABC and coop-pABC (b).

Table 7. Speedup and efficiency values for coop-pABC.

Evaluations	ABC and coop-pABC Algorithms	
	Speedup	Efficiency
1,000	2.58907	0.64726
2,000	2.51792	0.62948
10,000	2.32833	0.58208

Another comparison between sABC and parallel ABC algorithms can be made about the convergence characteristics of them illustrated in Fig. 4 below. When the convergence curves given in Fig. 4 are investigated, it is clearly seen that convergence performance of the coop-pABC algorithm is better than the convergence performances of the sABC and pABC algorithms. Although the initial mean best coverage values of parallel ABC algorithms is less than the initial mean best coverage value of sABC algorithm, they reached sABC algorithm before completion of the first $1,000$ evaluations and then start to produce more eligible mean best coverage values than sABC algorithm.

In order to make a visual investigation how the sensors are positioned by the sABC, pABC and coop-pABC algorithms and how the areas being covered change for the different termination conditions, the Figs. 5-10 should be utilized. As easily seen from the Figs. 5-10, successfully coveraged areas by the algorithms are rational with the total number of evaluations. With the completion of the $1,000$ evaluations, both serial and parallel implementations of the ABC algorithm produce deployments in which some sensors are located relatively close positions and coverage areas of them are overlapped. However, when the number of evaluations is set to $10,000$, overlapped sensors are scattered more robustly and coop-pABC algorithm outperforms sABC and pABC algorithms in terms of mean best coverage ratios.

Deciding whether coop-pABC algorithm can be interchangeable with the sABC or pABC algorithms, an information extracted from a statistical test should be utilized. For this purpose, a nonparametric test called Wilcoxon signed rank test is used with the significance level (p) less than 0.05. From the test results given in the Table 8 for $10,000$ fitness evaluations, it is seen that there is no significant difference between serial and parallel implementations of the ABC algorithm even though coop-pABC algorithm produces better mean best coverage values and parallel implementations can be used on behalf of sABC algorithm if the running environments are designed for utilizing the multi-core or multi-node based architectures.

Table 8. Statistical comparison between ABC algorithms.

Test statistics	sABC/coop-pABC	pABC/coop-pABC
Z-Value	-1.784925	-1.274946
p-Value	0.074274	0.202328
Sign.	-	-

6. Conclusion

In this study, ABC algorithm was parallelized for running on a multi-core processor and it performance was tested on solving wireless sensor deployment problem. Parallelized ABC algorithm by dividing the whole bee colony into subcolonies running simultaneously was powered with the cooperative emigrant creation approach and the results obtained with the mentioned ABC algorithm were compared to the results obtained with

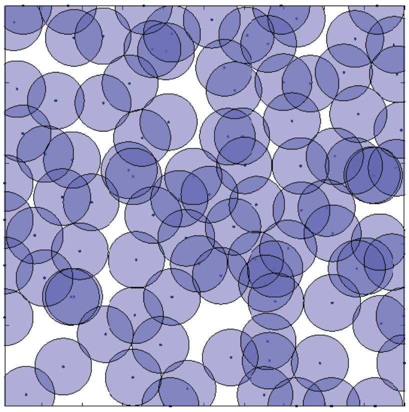


Figure 5. The best coverage of sABC for 1,000 evaluations

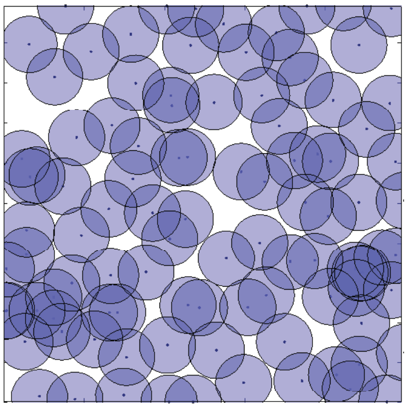


Figure 6. The best coverage of pABC for 1,000 evaluations

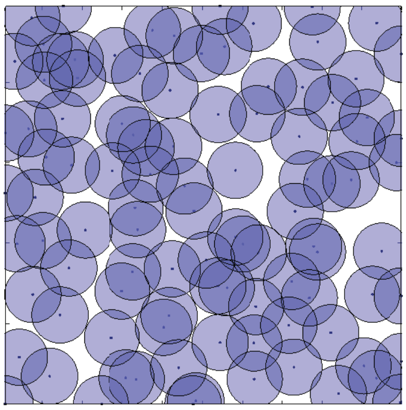


Figure 7. The best coverage of coop-pABC for 1,000 eveluations

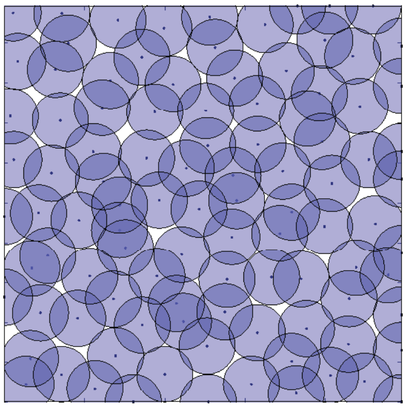


Figure 8. The best coverage of sABC for 10,000 evaluations

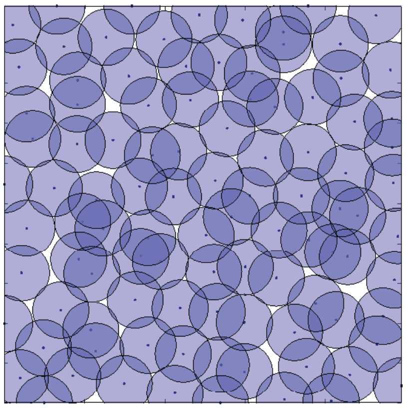


Figure 9. The best coverage of pABC for 10,000 evaluations

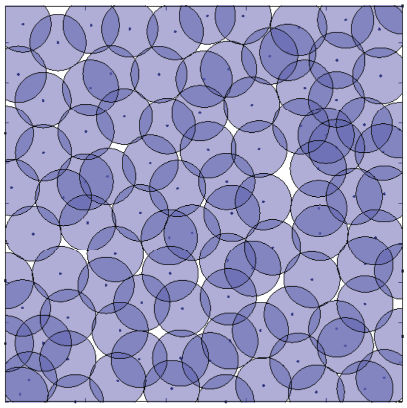


Figure 10. The best coverage of coop-pABC for 10,000 eveluations

standard serial and conventional parallel ABC algorithms. Comparative studies showed that cooperative model is still capable of increasing convergence speed and improving solution qualities of parallel ABC algorithm for sensor deployment problem as seen in the numerical benchmark problems by adding extra computational burden that changes directly with the migration period to the execution time of the algorithm.

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RESEARCH ARTICLE

An application of the MEFM to the modified Boussinesq equation

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ABSTRACT

In this paper, some travelling wave solutions of the Modified Boussinesq (MBQ) equation are obtained by using the modified expansion function method (MEFM). When the obtained solutions are commented, trigonometric functions including hyperbolic features are obtained. The 2D and 3D graphics of the solutions have been investigated by selecting appropriate parameters. All the obtained solutions provide the MBQ equation. In this work, all mathematical calculations are done with Wolfram Mathematica software.

1. Introduction

The solution of nonlinear partial differential equations has a measure in real life. For this reason, many methods have been developed and applied to solve these equations. Some of these, respectively the trial equation method [1], the new function methods [2-6], the extended trial equation method [7], Kudryashov method [8], the sine-Gordon expansion method [9-10] and so on. In this study, we apply the modified expansion function method (MEFM) [11-13] to solve a nonlinear MBQ equation and find new interactions among travelling wave solutions. Boussinesq–type equations of higher order in dispersion as well as in nonlinearity are reproduced for wave–current interaction over an unbalanced bottom. There are various methods in the literature to obtain the solution of the equation. Some of those; tanh method, the modified decomposition method and bilinearization method

In Section 2, Information about the modified expansion function method will be given.

In Section 3 the modified expansion function method is applied to the MBQ equation and the new exact wave solution to this problem is obtained. The 2D and 3D graphics of the solutions were drawed by using the Mathematica software program.

The modified Boussinesq equation can be defined as follows [14-16],

$$u_{tt} - u_{xxtt} - u_{xx} + \frac{a}{2}\left(u^2\right)_{xx} = 0. \tag{1}$$

2. Modified Expansion Function method

In this part, we will be given information about MEFM. Consider the following nonlinear partial differential equation (NPDE):

$$P\left(u, u^2, u_x, u_t, u_{xx}, u_{tt}, \left(u^2\right)_{xx}, u_{xxt}, u_{xxtt}\right) = 0, \tag{2}$$

where $u = u(x,t)$ is unknown function, P is a polynomial in $u(x,t)$ and its derivatives.

The general form of the nonlinear partial differential equation (2) is given above. By applying wave conversion to NPDE expression (3), the general form of the following nonlinear ordinary differential equation (4) is obtained.

Step 1: Consider the following travelling wave transformation:

$$u(x,t) = u(\xi), \quad \xi = v\left(x - ct\right). \tag{3}$$

Substituting Eq. (3) into Eq. (2), gives the following nonlinear ordinary differential equation (NODE);

$$N\left(u, u^2, \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \dots\right) = 0. \tag{4}$$

Step 2: We assume the following solution;

*Corresponding author

$$u(\xi)=\frac{\sum_{i=0}^{m} A_i[e^{-\vartheta(\xi)}]^i}{\sum_{j=0}^{n} B_j[e^{-\vartheta(\xi)}]^j}= \tag{5}$$

$$=\frac{A_0+A_1e^{-\vartheta}+\ldots+A_me^{-m\vartheta}}{B_0+B_1e^{-\vartheta}+\ldots+B_ne^{-n\vartheta}},$$

where $A_i, B_j, (0\le i\le m,\ 0\le j\le n)$.

m, n are positive integers that can be obtained by using the balancing principle.

$$\vartheta'(\eta)=e^{-\vartheta(\eta)}+ke^{\vartheta(\eta)}+\lambda. \tag{6}$$

Eq.(6) has the following families of solutions [17]:

Family 1: When, $k\neq 0, \lambda^2-4k>0$,

$$\vartheta(\eta)=ln\left(\frac{-\sqrt{\lambda^2-4k}}{2k}\tanh\left(\frac{\sqrt{\lambda^2-4k}}{2}(\eta+EE)\right)-\frac{\lambda}{2k}\right). \tag{7}$$

Family 2: When, $k\neq 0$, $\lambda^2-4k<0$,

$$\vartheta(\eta)=ln\left(\frac{\sqrt{-\lambda^2+4k}}{2k}\tan\left(\frac{\sqrt{-\lambda^2+4k}}{2}(\eta+EE)\right)-\frac{\lambda}{2k}\right). \tag{8}$$

Family 3: When, $k=0, \lambda\neq 0$, $\lambda^2-4k>0$,

$$\vartheta(\eta)=-ln\left(\frac{\lambda}{e^{\lambda(\eta+EE)}-1}\right). \tag{9}$$

Family 4: When, $k\neq 0, \lambda\neq 0$, $\lambda^2-4k=0$,

$$\vartheta(\eta)=ln\left(-\frac{2\lambda(\eta+EE)+4}{\lambda^2(\eta+EE)}\right). \tag{10}$$

Family 5: When, $k=0, \lambda=0$, $\lambda^2-4k=0$,

$$\vartheta(\eta)=ln(\eta+EE), \tag{11}$$

Where, EE is a integral constant.

Step 3: By substituting Eq. (5) and its derivatives into Eq. (4), we get algebraic equation system. This system was solved by using the Mathematica software program and then the solutions of the MBQ equation were obtained.

3. Application

In this section, the modified expansion function method will be used to obtain solutions of the MBQ equation. Consider the following travelling wave transformation:

$$u(x,t)=u(\xi),\quad \xi=v\left(x-ct\right). \tag{12}$$

the following nonlinear ordinary differential equation is obtained,

$$au^2+2\left(c^2-1\right)u-2c^2v^2u''=0. \tag{13}$$

If the balancing procedure is applied to equation (13), we get $n=m+2$ equality.

Choosing $m=1$, we get $n=3$. Eq. (5) for m and n values is obtained as follows;

$$u\left(\xi\right)=\frac{A_0+A_1e^{-\vartheta}+A_2e^{-2\vartheta}+A_3e^{-3\vartheta}}{B_0+B_1e^{-\vartheta}}. \tag{14}$$

If Eq. (14) is regulated according to the necessary term in equation (13), then the following system of algebraic equations is obtained which consists of the coefficients of $e^{-\vartheta(\xi)}$.

Some suitable coefficients obtained by using the Mathematica package program are given below.

Case-1:

$$A_0=\frac{12\mu v^2B_0}{a-a\left(\lambda^2-4\mu\right)v^2},$$

$$A_1=-\frac{12v^2\left(\lambda B_0+\mu B_1\right)}{a\left(-1+\left(\lambda^2-4\mu\right)v^2\right)},$$

$$A_2=-\frac{12v^2\left(B_0+\lambda B_1\right)}{a\left(-1+\left(\lambda^2-4\mu\right)v^2\right)},$$

$$A_3=\frac{12v^2B_1}{a-a\left(\lambda^2-4\mu\right)v^2},$$

$$c=\frac{1}{\sqrt{1-v^2\left(\lambda^2-4\mu\right)}}.$$

Substituting these coefficients into Eq. (14), the following solutions:

Family 1: When, $k\neq 0, \lambda^2-4k>0$, solution of equation (1),

$$u_1\left(x,t\right)=\frac{\left(12\left(\lambda^2-4\mu\right)\mu v^2\right)}{\left(a\left(-1+\left(\lambda^2-4\mu\right)v^2\right)\left(\lambda Cosh\left[\psi\right]+\sqrt{\lambda^2-4\mu}Sinh\left[\psi\right]\right)^2\right)}, \tag{15}$$

where,

$$\left(\psi=\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(EE+\xi\right)v\right]\right).$$

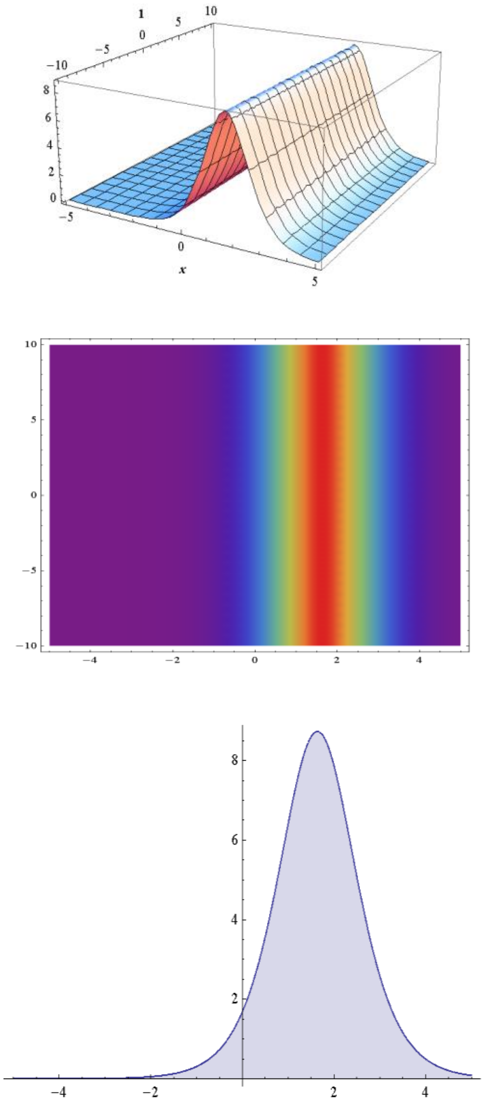

Figure-1. The 3D, density graphic and 2D surfaces of Eq. (15) in $\lambda = 2, \mu = 0.2, c = 4, a = 0.5, \ \nu = 1, EE = 0.75$ and $t = 1$

Family 2: When, $k \neq 0$, $\lambda^2 - 4k < 0$, we get,

$$u_2(x,t) = \frac{\left(12\left(\lambda^2 - 4\mu\right)\mu\nu^2\right)}{\left(a\left(-1+\left(\lambda^2-4\mu\right)\nu^2\right)\left(\lambda Cos[\upsilon] - \sqrt{-\lambda^2+4\mu}Sin[\upsilon]\right)^2\right)}, \tag{16}$$

where,

$$\left(\upsilon = \left[\frac{1}{2}\sqrt{-\lambda^2+4\mu}\left(EE+\xi\right)\nu\right]\right).$$

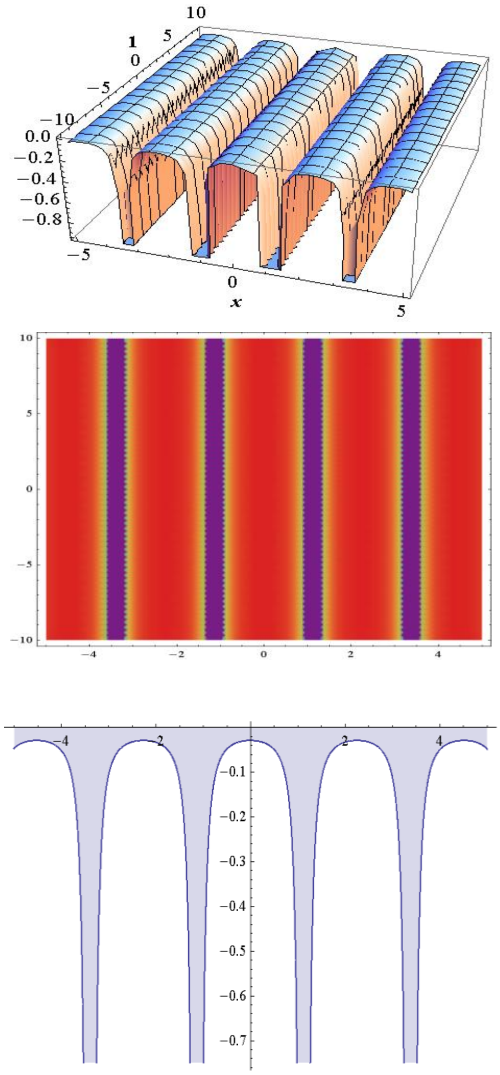

Figure-2. The 3D, density graphic and 2D surfaces of Eq. (16) in $\lambda = 2, \mu = 0.2, c = 4, a = 0.5, \ \nu = 1, EE = 0.75$ and $t = 1$.

Family 3: $k = 0, \lambda \neq 0$, $\lambda^2 - 4k > 0$,

$$u_3(x,t) = \left(-\frac{3\lambda^2\nu^2 Csch\left[\frac{1}{2}\lambda\left(EE+\left(\xi\right)\nu\right)\right]^2}{-a+a\lambda^2\nu^2}\right), \tag{17}$$

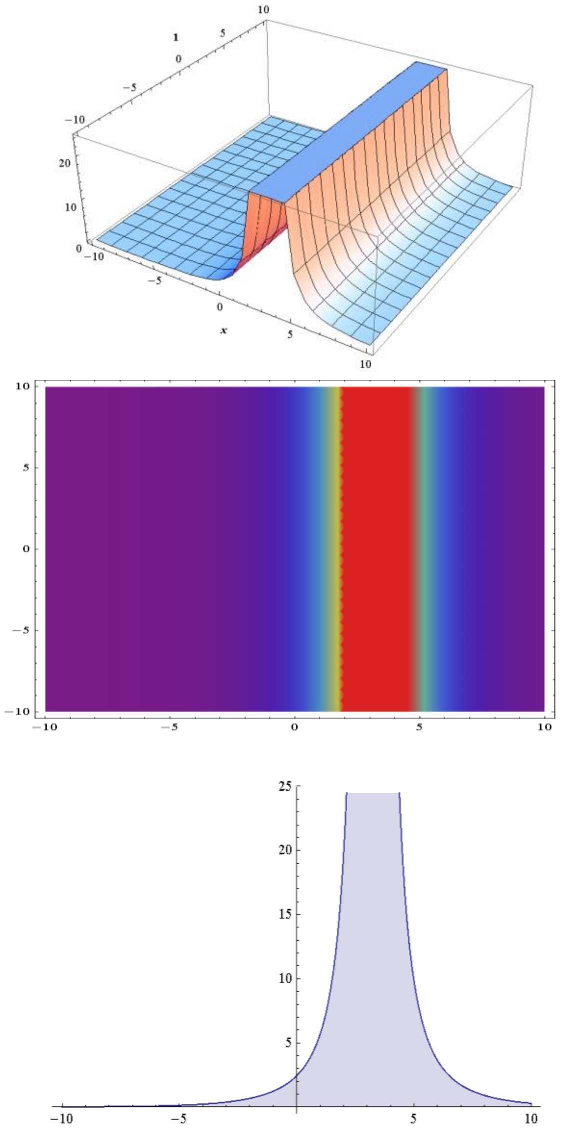

Figure-3. The 3D, density graphic and 2D surfaces of Eq. (17) in $\lambda = 0.5, \mu = 0, c = 4, a = 0.5, \nu = 1, EE = 0.75$ and $t = 1$.

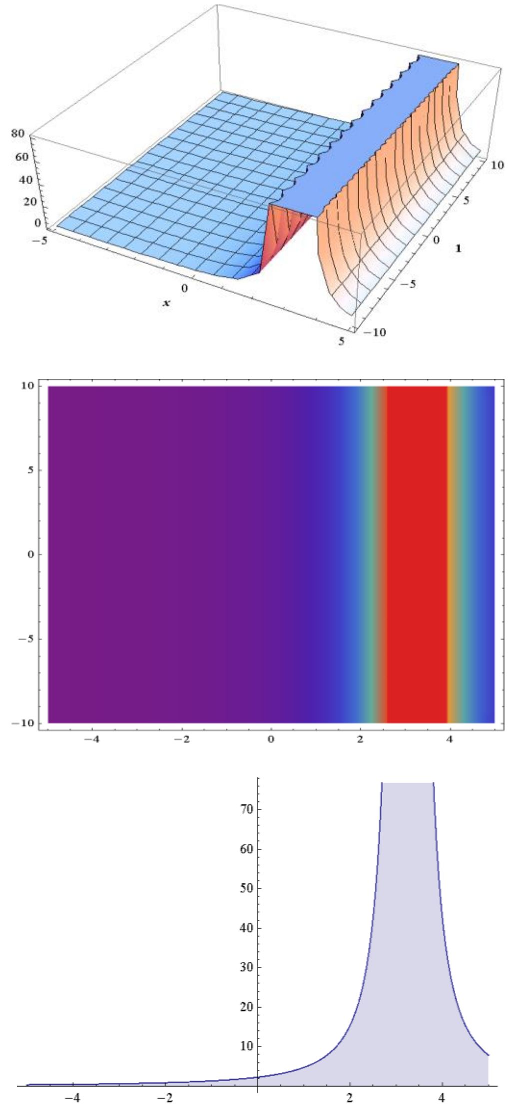

Figure-4. The 3D, density graphic and 2D surfaces of Eq. (18) in $\lambda = 0, \mu = 0, c = 4$ $a = 0.5, \nu = 1, EE = 0.75$ and $t = 1$.

According to Family-4, the solution does not exist.

Family 5: When $k = 0$, $\lambda = 0$ and $\lambda^2 - 4k = 0$,

$$u_5(x,t) = \left(\frac{12\nu^2}{a\left(EE + (\xi)\nu\right)^2} \right), \tag{18}$$

Case-2:

$$A_0 = \frac{\left(-1+c^2\right)\left(-\sqrt{c^4\left(\lambda^2-4\mu\right)^2}+c^2\left(\lambda^2+8\mu\right)\right)B_0}{a\sqrt{c^4\left(\lambda^2-4\mu\right)^2}},$$

$$A_1 = \frac{\left(\left(-1+c^2\right)\left(12c^2\lambda B_0+\left(-\sqrt{c^4\left(\lambda^2-4\mu\right)^2}+c^2\left(\lambda^2+8\mu\right)\right)B_1\right)\right)}{a\sqrt{c^4\left(\lambda^2-4\mu\right)^2}},$$

$$A_2 = \frac{\left(12c^2\left(-1+c^2\right)\left(B_0+\lambda B_1\right)\right)}{a\sqrt{c^4\left(\lambda^2-4\mu\right)^2}},$$

$$A_3 = \frac{\left(12c^2\left(-1+c^2\right)B_1\right)}{a\sqrt{c^4\left(\lambda^2-4\mu\right)^2}},$$

$$\nu = -\frac{\sqrt{-1+c^2}}{\left(\sqrt{c^4\left(\lambda^2-4\mu\right)^2}\right)^{\frac{1}{4}}}.$$

Substituting these coefficients into Eq. (14), the following solutions:

Family 1: When, $k \neq 0$, $\lambda^2 - 4k > 0$, we get

$$u_1(x,t) = \frac{\left(-1+c^2\right) Sech[\psi]^2 \left(-2\left(5c^2(\chi)+\sqrt{c^4(\chi)^2}\right)\mu\right)}{\left(a\sqrt{c^4(\chi)^2}\left(\lambda+\sqrt{\chi}Tanh[\psi]\right)^2\right)}, \tag{19}$$

where,

$$\left(\chi = \lambda^2 - 4\mu, \quad \psi = \sqrt{\lambda^2-4\mu}Tanh\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(EE+\xi\right)\right]\right),$$

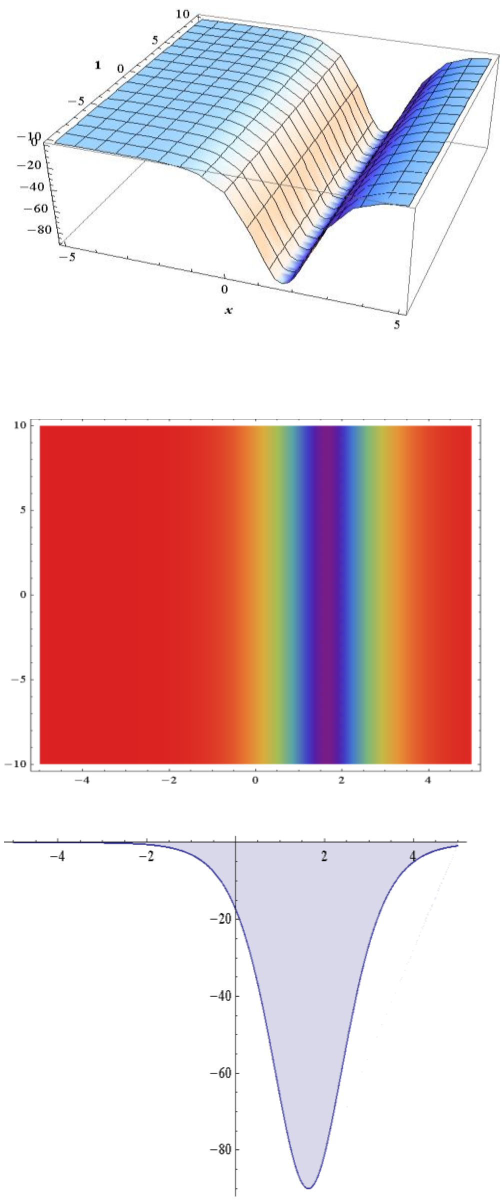

Figure-5. The 3D, density graphic and 2D surfaces of Eq. (19) in $\lambda = 0.2, \mu = 2, c = 4$ $a = 0.5, \nu = 1, EE = 0.75$ and $t = 1$.

Family 2: When, $k \neq 0$, $\lambda^2 - 4k < 0$,

$$u_2(x,t) = \frac{-\left(-1+c^2\right)\left(\sqrt{c^4\chi^2}\left(\lambda-\sqrt{-\chi}Tan[\tau]\right)^2\right) + c^2\chi\left(-\lambda^2+12\mu+2\lambda\sqrt{-\chi}Tan[\tau]\right)}{\left(\left(\lambda^2+8\mu\right)Tan[\tau]^2\right)}, \tag{20}$$

where,

$$\left(\tau = \frac{1}{2}\sqrt{-\chi}\left(EE+\xi\right)\nu\right).$$

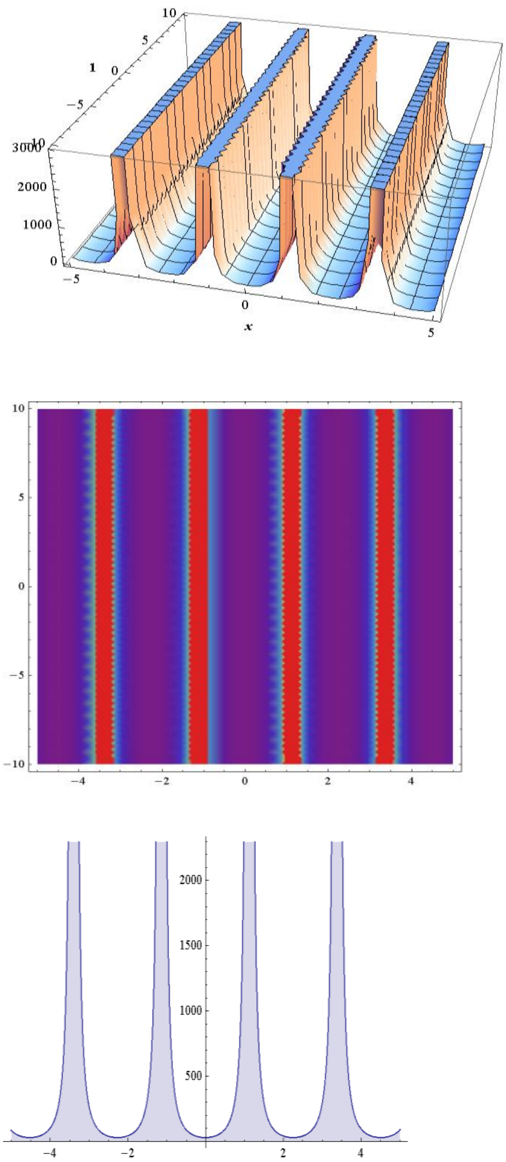

Figure-6. The 3D, density graphic and 2D surfaces of Eq. (20) in $\lambda = 0.5, \mu = 2, c = 4$ $a = 0.5, \nu = 1, EE = 0.75$ and $t = 1$.

Family 3: $k = 0, \lambda \neq 0$, $\lambda^2 - 4k > 0$,

$$u_3(x,t) = \frac{\left(-1+c^2\right)\left(-\sqrt{c^4\lambda^4}+c^2\lambda^4\left(1+3Csch\left[\frac{1}{2}\lambda\left(EE+\xi\right)\nu\right]^2\right)\right)}{a\sqrt{c^4\lambda^4}}, \tag{21}$$

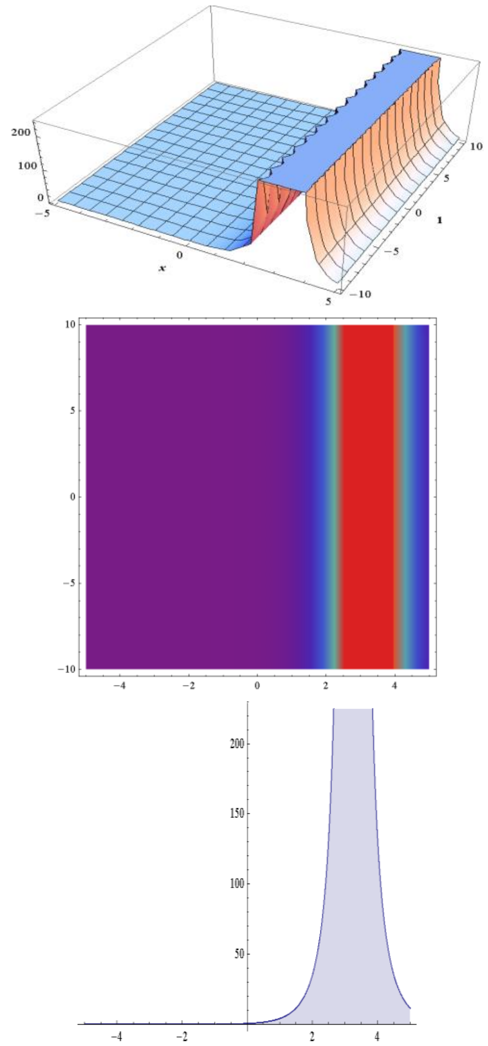

Figure-7. The 3D, density graphic and 2D surfaces of Eq. (21) in $\lambda = 0, \mu = 2, c = 4$ $a = 0.5, \nu = 1, EE = 0.75$ and $t = 1$.

Family-4 and Family-5, the solution does not exist.

4. Conclusion

In this study, we obtained some travelling wave solutions of Boussinesq equation by using modified expansion function method. The results show that the modified expansion function method is a suitable mathematical method for solving nonlinear partial differential equations. The resulting solutions were checked with the Mathematica software. These solutions have been obtained by MEFM for the first time in the literature.

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RESEARCH ARTICLE

On the numerical investigations to the Cahn-Allen equation by using finite difference method

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ABSTRACT

In this study, by using the finite difference method (FDM for short) and operators, the discretized Cahn-Allen equation is obtained. New initial condition for the Cahn-Allen equation is introduced, considering the analytical solution given in *Application of the modified exponential function method to the Cahn-Allen equation, AIP Conference Proceedings 1798, 020033 [1]*. It is shown that the FDM is stable for the usage of the Fourier-Von Neumann technique. Accuracy of the method is analyzed in terms of the errors in L_2 and L_∞. Furthermore, the FDM is treated in order to obtain the numerical results and to construct a table including numerical and exact solutions as well as absolute measuring error. A comparison between the numerical and the exact solutions is supported with two and three dimensional graphics via Wolfram Mathematica 11.

1. Introduction

Russel has firstly studied the solitary wave [2,4] by following the water wave travelling through a tube. Investigation of the analytical and numerical solutions as well as other studies to the various class of nonlinear partial differential equations play an important role in the field of nonlinear sciences.

Most recently, some serious methods have been developed in order to solve nonlinear differential equation. For example, (G'/G)-expansion method [5,6], the improved (G'/G)-expansion method [7-9], the modified simple equation method [10], the Sumudu transform method [11-14], the Bäcklund transform method [15], the homotopy analysis method [16,17], the exponential function method [18-20], the modified exponential function method [21], generalized Bernoulli sub-ODE method [22], improved Bernoulli sub-ODE method [24-26], weak solutions[27] and galerkin method [28].

In the current work, we consider the Cahn-Allen equation given as:

$$u_t = u_{xx} - u^3 + u. \tag{1}$$

By using first integral method, Bulut et al. [23] have obtained some soliton to Eq. (1).

The discretize equation to the Cahn-Allen equation is derived by using the finite difference method (FDM) and its operators. We observe that the numerical method is stable with the Eq. (1) is stable when the Fourier-Von Neumann technique is utilzed. Furthermore, the accuracy in terms of the errors in and is analyzed. We then utilized the FDM in approximating exact and numerical solutions to Eq. (1). We present the computed exact and numerical approximations as well as the absolute error in tables. We compare the exact and numerical approximations calculated and support the comparison with some graphics plots, which are sketched by using the Wolfram Mathematica 11.

2. Fundamental properties of methods

2.1 Analysis of FDM

Some important notations are needed in order to describe the finite forward difference method, these are:

- Δx, which is the spatial step
- Δt, which is the time step

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- $x_i = a + i\Delta x,\ i = 0,1,2,\ldots,N$ points, which are the coordinates of mesh and $N = \frac{b-a}{\Delta x},\ t_j = j\Delta t,\ j = 0,1,2,\ldots,M$ and $M = \frac{T}{\Delta t}$.
- The function $u(x,t)$ is the value of the solution at $u(x_i,t_j) \cong u_{i,j}$ (grid points), where $u_{i,j}$ will is the numerical approximations of the exact value of $u(x,t)$ at the points (x_i,t_j).

The difference operators are given as follows:

$$H_t u_{i,j} = u_{i,j+1} - u_{i,j}, \tag{2}$$

$$H_{xx} u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}. \tag{3}$$

Thus, the derivatives involve in Eq. (1) can be given in finite difference operators form as

$$\left.\frac{\partial u}{\partial t}\right|_{i,j} = \frac{H_t u_{i,j}}{\Delta t} + O(\Delta t^2), \tag{4}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{H_{xx} u_{i,j}}{(\Delta x)^2} + O(\Delta x^2). \tag{5}$$

The difference operator form to Eq. (1) is given as

$$\frac{H_t u_{i,j}}{\Delta t} = \frac{H_{xx} u_{i,j}}{(\Delta x)^2} - \left(u_{i,j}\right)^3 + u_{i,j}. \tag{6}$$

Inserting Eq. (4) and (5) into Eq. (1), one can be written as indexed

$$\begin{aligned} u_{i+1,j} = -u_{i-1,j} + u_{i,j}\left(2 - (\Delta x)^2 - \frac{(\Delta x)^2}{\Delta t}\right) \\ + (\Delta x)^2 (u_{i,j})^3 + \frac{(\Delta x)^2}{\Delta t} u_{i,j+1}, \end{aligned} \tag{7}$$

where the initial values $u_{i,0} = u_0(x_i)$.

2.2. Consistency analysis

In this subsection, the consistency of Eq. (1) with difference method is discussed. Firstly, the Taylor series expansions as taking the following form [11-13],

$$u_{i,j+1} = u_{i,j} + \Delta t \frac{\partial u}{\partial t} + O(\Delta t)^2, \tag{8}$$

$$u_{i-1,j} = u_{i,j} - \Delta x \frac{\partial u}{\partial x} + (\Delta x)^2 \frac{\partial^2 u}{\partial x^2} - O(\Delta x^3). \tag{9}$$

One may define the operator L as

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}.$$

The indexed form of operator L takes the following form:

$$L_{i,j} = \frac{H_t u_{i,j}}{\Delta t} - \frac{H_{xx} u_{i,j}}{(\Delta x)^2}. \tag{10}$$

Inserting the indexed form (8) and (9) into the equality (10) and making some theoretical calculations, then the approach will be the $\Delta t \to 0$ and $\Delta x \to 0$. Therefore, the equality (10) will be same as left hand side of the Eq. (1). Thus, it can be seen that the Eq. (1) is consistent with FDM.

2.3 Truncation error and stability analysis

In this subsection, the stability and error analysis for the FDM are studied. For the stability, if there is a perturbation in the initial condition and then the small change would not cause the large error in the numerical solution. Simply, stability means that the scheme does not amplify errors and the error caused by a small perturbation in the numerical solution remains bound.

Theorem 1. The truncation error of the finite different method to the Eq. (1) is $(\Delta x)^2\left[O(\Delta t)^2 + O(\Delta x)^3\right]$.

Proof. Inserting Eq. (4) and (5) into Eq. (1) gives

$$\begin{aligned} \frac{H_t u_{i,j}}{\Delta t} + O(\Delta t)^2 = \left(\frac{H_{xx} u_{i,j}}{(\Delta x)^2} + O(\Delta x)^3\right) \\ -\left(u_{i,j}\right)^3 + u_{i,j}. \end{aligned} \tag{11}$$

Inserting the equalities (2) and (3) into the Eq. (11) and do some necessary manipulations, then we obtain the following equality

$$\begin{aligned} u_{i+1,j} = -u_{i-1,j} + u_{i,j}\left(2 - (\Delta x)^2 - \frac{(\Delta x)^2}{\Delta t}\right) \\ + (\Delta x)^2 (u_{i,j})^3 + \frac{(\Delta x)^2}{\Delta t} u_{i,j+1} \\ + (\Delta x)^2 (O(\Delta t)^2 + O(\Delta x)^3). \end{aligned} \tag{12}$$

Utilizing Eq. (12), one may write numerical solution $\hat{U}$ as

$$\begin{aligned} \hat{U} = -u_{i-1,j} + u_{i,j}\left(2 - (\Delta x)^2 - \frac{(\Delta x)^2}{\Delta t}\right) \\ + (\Delta x)^2 (u_{i,j})^3 + \frac{(\Delta x)^2}{\Delta t} u_{i,j+1}, \end{aligned}$$

and the truncation error E as

$$E = (\Delta x)^2\left[O(\Delta t)^2 + O(\Delta x)^3\right].$$

Moreover, if Δt and Δx are considered as small as necessary, truncation error will be obviously very small. The limit of E can be written as

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} E = 0 \cdot$$

We can see that if Δt and Δx are configured for a value close to zero $\delta > 0$, the following inequality is gotten

$$|E| < \delta ,$$

which proves the stability of the FDM.

Theorem 2. The FDM in respect to the Cahn-Allen equation is unconditionally stable.

Proof. We consider the Von Neumann's Stability of the finite difference method for the Cahn-Allen. Let

$$u_{i,j} = u(i\Delta x, j\Delta t) = u(p,q) = \varepsilon^q e^{I\xi p}, \xi \in [-\pi, \pi], \tag{13}$$

where $p = i\Delta x,\ q = j\Delta t$ and $I = \sqrt{-1}$. Inserting Eq. (2), (3) and (13) into the equality (6), we can obtain

$$\varepsilon \to 0,$$

According to the Von Neumann's Stability analysis [29], the FDM is stable if $|\varepsilon| \leq 1$. Hence, the FDM is unconditionally stable with the Cahn-Allen equation.

2.4. L_2 and L_∞ Error Norms

To show how close the numerical approximations are close to the exact approximations the L_2 and L_∞ error norms are utilized [30].

The L_2 error norm is defined as [30].

$$L_2 = \left\| u^{exact} - u^{numeric} \right\|_2 = \sqrt{h \sum_{j=0}^{N} \left| u_j^{exact} - u_j^{numeric} \right|^2},$$

and L_∞ error norm is defined as [30]

$$L_\infty = \left\| u^{exact} - u^{numeric} \right\|_\infty = \underset{j}{Max} \left| u_j^{exact} - u_j^{numeric} \right|.$$

3. Application

In this section, we apply Finite Difference Method for Eq. (1) and consider numerical experiments. Recall the following hyperbolic function solution for Eq. (1) given in [1]:

$$u_1(x,t) = -\frac{\left(3A_1 + \sqrt{9A_1^2 + 24cA_0B_1}\right)\left(-1 + Tanh[f(x,t)]\right)}{6A_1 + 2\sqrt{9A_1^2 + 24cA_0B_1} - 6B_1\left(1 + Tanh[f(x,t)]\right)}, \tag{14}$$

where $f(x,t) = \dfrac{3c_1 - 3ct + \sqrt{2}cx}{4c}$ and

$\mu \neq 0,$

$$\frac{\left(3A_1 - 3B_1 + \sqrt{9A_1^2 + 24cA_0B_1}\right)^2}{4c^2B_1^2} - \frac{3\left(4cA_0B_1 + (A_1 - B_1)\left(3A_1 + \sqrt{9A_1^2 + 24cA_0B_1}\right)\right)}{2c^2B_1^2} > 0.$$

If we put

$c = 0.6,\ A_0 = -3,\ B_1 = -5,\ A_1 = -1, c_1 = 0.1,$

$0 < x < 1$ and $0 < t < 1$ for Eq. (14), the initial condition is

$$u_0(x) = u(x,0) = -\frac{\left(3A_1 + \sqrt{9A_1^2 + 24cA_0B_1}\right)\left(-1 + \text{Tanh}\left[\frac{3c_1 + \sqrt{2}cx}{4c}\right]\right)}{6A_1 + 2\sqrt{9A_1^2 + 24cA_0B_1} - 6B_1\left(1 + \text{Tanh}\left[\frac{3c_1 + \sqrt{2}cx}{4c}\right]\right)}, \tag{15}$$

and under the above assumptions the exact solution of the Eq. (1) is as following

$$u(x,t) = -\frac{12\left(-1 + \text{Tanh}[0.416667(0.3 - 1.8t + 0.848528x)]\right)}{24 + 30\left(1 + \text{Tanh}[0.416667(0.3 - 1.8t + 0.848528x)]\right)}. \tag{16}$$

Eq. (1) can be written as indexed with the help of finite difference operators

$$u_{i+1,j} = -0.0001\left[10000\, u_{i-1,j} - 19999\, u_{i,j} - u_{i,j}^3 - 100\left(-u_{i,j} + u_{i,j+1}\right)\right].$$

A comparison of the obtained exact and numerical solutions are tabulate in Table 1.

Table 1. Numerical and exact solutions of equation (1) and absolute errors when $\Delta x = 0.01$.

x_i	t_j	*Numerical solution*	*Exact Solution*	*Absolute Error*
0.00	0.01	0.184247	0.184258	1.06626×10^{-5}
0.01	0.01	0.183187	0.183197	1.06500×10^{-5}
0.02	0.01	0.182131	0.182142	1.06370×10^{-5}
0.03	0.01	0.181080	0.181091	1.06236×10^{-5}
0.04	0.01	0.180034	0.180044	1.06098×10^{-5}
0.05	0.01	0.178992	0.170900	1.05956×10^{-5}
0.06	0.01	0.177955	0.177966	1.05810×10^{-5}

Table 2. L_2 and L_∞ error norm when $0 \leq h \leq 1$ and $0 \leq x \leq 1$

$\Delta x = \Delta t$	L_2	L_∞	0.2
2.01978×10^{-3}		4.317×10^{-3}	0.1
6.96142×10^{-4}		1.074×10^{-3}	
0.05	2.42301×10^{-4}	2.670×10^{-4}	
0.01	1.04962×10^{-5}	1.100×10^{-5}	

Table2 shows that when Δx and Δt are small, the L_2 and L_∞ error norm are decreasing. From Table 1-2 it is easily seen that results are in good agreement with the exact solution.

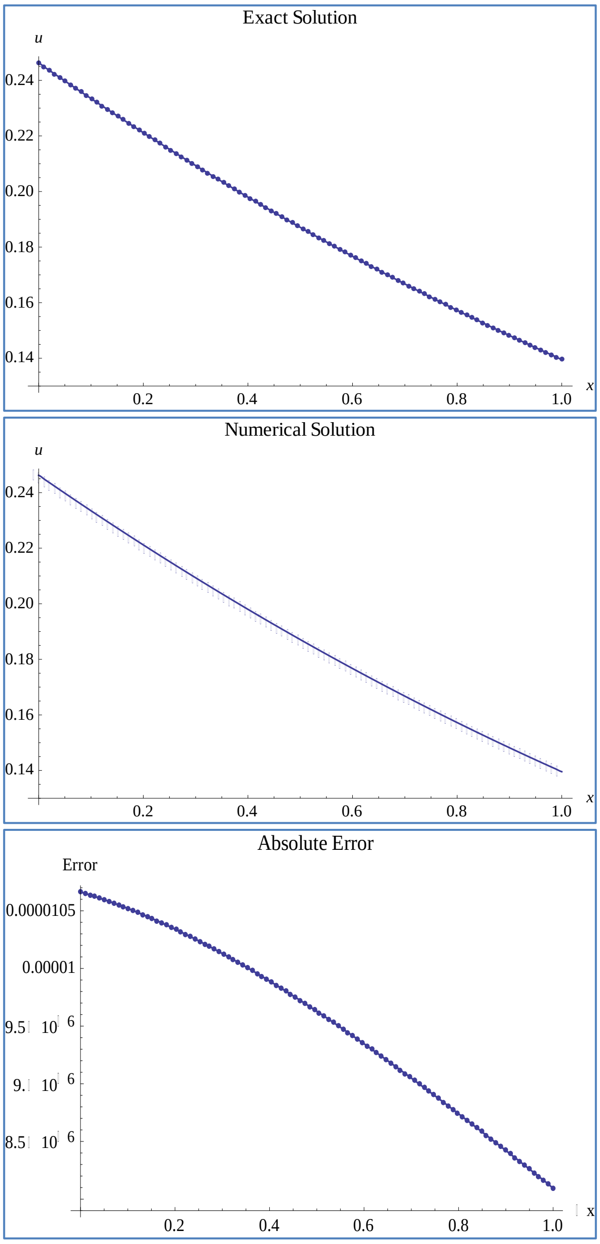

Figure 1. Numerical solution of Eq. (1) for finite difference method

Fig. 1 displays the physical behavior of the solution and shows that the exact approximations values are almost close to the numerically computed values. It is known that the truncation error depends on the choice of Δx and Δt. Choosing the values to be very small gives rise to very small truncation error. This behavior of the numerical and exact solutions can be seen in the graphs above when the values of $\Delta x = \Delta t = 0.01$.

4. Remark

The numerical results for example 1 have been obtained by using the programming language Wolfram Mathematica package. To the best of our knowledge, these numerical solutions have not been published previously, and these results are new numerical solutions for (1).

5. Conclusion

In this study, the FDM is used in approximating the numerical solutions to the Cahn-Allen equation. FDM is a useful numerical scheme for approximating the solutions of various nonlinear differential equations by defining suitable differential operators. The initial condition for the Cahn-Allen equation is obtained using the new analytical solution. The Cahn-Allen equation is written as indexed with the help of finite difference operators. Error analysis of the index equation was analyzed. Cahn-Allen equation is discussed with an example and error estimates obtained for the FDM. Furthermore, the behavior of potentials *u* and absolute error are examined graphically.

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RESEARCH ARTICLE

Optimal control analysis of deterministic and stochastic epidemic model with media awareness programs

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ABSTRACT

The present study considered the optimal control analysis of both deterministic differential equation modeling and stochastic differential equation modeling of infectious disease by taking effects of media awareness programs and treatment of infectives on the epidemic into account. Optimal media awareness strategy under the quadratic cost functional using Pontrygin's Maximum Principle and Hamiltonian-Jacobi-Bellman equation are derived for both deterministic and stochastic optimal problem respectively. The Hamiltonian-Jacobi-Bellman equation is used to solve stochastic system, which is fully non-linear equation, however it ought to be pointed out that for stochastic optimality system it may be difficult to obtain the numerical results. For the analysis of the stochastic optimality system, the results of deterministic control problem are used to find an approximate numerical solution for the stochastic control problem. Outputs of the simulations shows that media awareness programs place important role in the minimization of infectious population with minimum cost.

1. Introduction

Epidemiology is the study of the spread of diseases with the objective to trace factors, which are responsible for or contribute to their occurrence. Mathematical modeling of the spread of infectious diseases continues to become an important tool in understanding the dynamics of diseases and in decision making processes regarding diseases intervention programs for disease in many countries. Controlling infectious diseases has been an increasingly complex issue in recent years. Media awareness program is an important strategy for the elimination of infectious diseases [1, 2]. The field of stochastic modeling of biological and ecological systems [3] is currently undergoing considerable development as of complex stochastic models by simulation methods are more feasible. Mathematicians have contributed a range of papers which can be found in the literature of probability theory and statistical physics characterizing the theoretical properties of a large variety of stochastic models.

Optimal control theory has found wide-ranging applications in biological and ecological problems. Specifically, there have been various studies of epidemiological models, where optimal control methods have been applied [4, 5]. Optimal control theory is a systematic approach to controller design where by the desired performance objectives are encoded in a cost function, which is subsequently optimized to determine the desired controller [6]. There are two underlying and universal themes i.e., dynamic programming and filtering. Dynamic programming is one of the fundamental tool of optimal control, the other being Pontryagins principle. Dynamic programming is a means by which candidates optimal control can be verified optimally. The procedure is to find a suitable solution to dynamic programming equation (DPE), which denotes the optimal performance and to use it to compare the performance candidates control may be determined from Pontryagins Maximum Principle [7] and later developed by Fleming and Rishel [8] is successfully applied

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in a number of studies, to explore optimal control theory in some mathematical models for infectious diseases. Epidemic models are inevitably affected by environmental white noise, which is an important component in realism, because it can provide an additional degree of realism in comparison to their deterministic counterparts. Many stochastic model for epidemic populations have been developed in literature [9,10]. For SDE models in epidemiology, optimal control has not been studied extensively. One of the reasons for this could very well be the difficulty with high dimensionality of the resulting partial differential equation (PDE) for the value function (See, Sulem and Tapiero [11]), for instance, a four-compartmental SIVR model such as in [12, 13] could easily lead to a PDE having the time variable together with three state variables. In control problems, the aim of the study is to characterize the control variable on a finite time interval, which minimizes the number of infected individuals balanced against the cost of controlling the epidemic.

In the present study it is proposed and developed optimal control policies for deterministic and stochastic SIR epidemic model with awareness programs by media. The aim of this model is to depict how the provision of awareness modifies the contact structure and thereby affects the future course of an epidemic. In the absence of any pharmaceutical intervention, to control the spread of disease at the population level needs to change the individual activities, which in turn depends on information being provided to the individuals about the epidemic. If the susceptibles are aware about the preventive measures for emergent disease, they are likely to modify their activities. The study contracts on disease which spread through interaction between susceptible and infective, i.e. direct contact. Therefore to control the outbreak of any epidemic, it is informed to avoid contact, by which some can contract infection and minimize the possibility of contracting infection. In vision of this, it is assumed that when awareness is propagated by media about the disease, susceptible form a separate class within the population i.e., to avoid being in contact with other members of the population. Another important aspect of this study is to check whether size of the infectious population is directly proportional to awareness campaigns by media. The explicit inclusion of awareness campaigns by media in the modeling process are assumed to be proportional to the size of infectious individuals in the population. This study differs from other epidemic modeling by performing the stochastic optimal control analysis, which is rarely studied by researchers like [6, 14] in the field of epidemics and by including the transmission of infection in two modes in the model, A.K. Misra [1], has discussed the epidemic model with media awareness and stability analysis for deterministic model by considering single transmission parameter β, in the present study, the transmission of infection is considered by two modes i.e. transmission between unaware susceptible and infectives and the transmission between hospitalized individuals and unaware susceptible denoted by by β_1 and β_2 respectively. It is assumed that the rate of contact of susceptible with infectives who are on treatment is much less than the infectives who are not on treatment ($\beta_2 \ll \beta_1$). This is so because on hospitalization of infectives for treatment their contact with susceptible group of a population is reduced and may contribute little to the spread of infection. In the numerical analysis of the deterministic and corresponding stochastic model, it is discussed the comparison of deterministic and stochastic solution and also shown, how the control variable vary for different values of a parameters. The rest of the work is organized as follows: Section 2 deals with deterministic model framework and optimal control analysis, while in Section 3 formulation of stochastic model with constant controls and optimal control analysis is carried out. Section 4, consist of numerical simulations and discussion of results and principle findings of the paper are discussed in Section 5.

2. Deterministic Model

In this section, deterministic nonlinear SIR model is considered by taking media awareness and treatment into account. The variables and parameters of the model are described in Table 1 and Table 2 respectively.

Table 1. Description of variables of the model.

Variables	Explanation
$X(t)$	The number of susceptible at time t;
$Y(t)$	The number of infectives at time t;
$X_m(t)$	The number of aware susceptible at time t;
$T(t)$	The infectives who are on treatment at time t;
$Z(t)$	The recovered population at time t;
$M(t)$	The cumulative density of awareness programs driven by media in the region at time t;

To model the situation considered a region with total population $N(t)$ at any instant of time t. By taking into account the aforementioned considerations, the system of equations that capture the dynamics of the infectious disease is designed and the ordinary differential equations of the system (1) is as follows.

Table 2. Description of parameters of the model

Parameters	Description
β_1	The contact rate of susceptible with infectives;
β_2	The contact rate of susceptible with hospitalized infectives;
Q	The constant rate of immigration of susceptible;
π	The dissemination rate of awareness among susceptible due to which they form a separate group;
π_0	The rate of transfer of aware susceptible to susceptible;
γ	The recovery rate;
δ	The disease induced death rate;
δ_1	The natural death rate from each class;
σ_1	The modification parameter due to treatment for recovery;
σ_2	The modification parameter for disease induced death rate due to treatment;
μ	The rate at which awareness programs has being implemented;
μ_0	The depletion rate of awareness programs due to infectiveness, social problems in population;
γ_0	The loss rate of immunity of recovered individuals;
ϕ	The rate at which infective are hospitalized for treatment;

$$\begin{aligned}
\frac{dX}{dt} &= Q - \beta_1 XY - \beta_2 XT - \pi XM + \pi_0 X_m \\
&\quad + \gamma_0 Z - \delta_1 X \\
\frac{dY}{dt} &= \beta_1 XY + \beta_2 XT - (\delta + \gamma + \phi + \delta_1)Y \\
\frac{dX_m}{dt} &= \pi XM - \pi_0 X_m - \delta_1 X_m \qquad (1)\\
\frac{dT}{dt} &= \phi Y - \sigma_1 \gamma T - \delta_1 T - \sigma_2 \delta T \\
\frac{dZ}{dt} &= \gamma Y + \sigma_1 \gamma T - \gamma_0 Z - \delta_1 Z \\
\frac{dM}{dt} &= \mu(Y+T) - \mu_0 M
\end{aligned}$$

where, $X > 0, Y > 0, X_m \geq 0, T \geq 0, Z \geq 0$ and $M \geq 0$.

To show the existence of the feasible set of a system (1) which attracts all solutions initiation in the interior of positive orthant, it has to prove that the system (1) is dissipative, i.e., all solutions are uniformly bounded in a proper subset $\Omega \in \Re^6_+$. Let $(X, Y, X_m, T, Z, M) \in \Re^6_+$ be any solution with non-negative initial conditions. By adding first five equations of system (1) it is obtained

$$\begin{aligned}
\frac{dN}{dt} &= Q - \delta Y - \sigma_2 \delta T - \delta_1 N \\
&\leq Q - \delta_1 N \qquad (2)
\end{aligned}$$

After solving equation (2), we have

$$N(t) \leq N(0)e^{-\delta_1 t} + \frac{Q}{\delta_1}(1 - e^{-\delta_1 t}) \qquad (3)$$

where $N(0)$ is the sum of initial values $X(0)$, $Y(0)$, $X_m(0)$, $T(0)$, $Z(0)$. Now from equation (3) as $\lim t \to \infty$, $N \to \frac{Q}{\delta_1}$, then $\frac{Q}{\delta_1}$ is the upper bound of N. Also from last equation of system (1), it is shown

$$\begin{aligned}
\frac{dM}{dt} &= \mu(Y+T) - \mu_0 M \\
\frac{dM}{dt} &\leq \frac{\mu Q}{\delta_1} - \mu_0 M \qquad (4)\\
\Rightarrow 0 &< M(t) \leq M(0)e^{-\mu_0 t} + \frac{\mu Q}{\mu_0 \delta_1}(1 - e^{-\mu_0 t})
\end{aligned}$$

and above result that $\frac{Q}{\delta_1}$ is the upper bound of N it can deduced that $\lim t \to \infty$ $M \to \frac{\mu Q}{\mu_0 \delta_1}$. Therefore the region of attraction is given by the set:

$$\begin{aligned}
\Omega = \Big\{ &(X,\ Y,\ X_m,\ T,\ Z,\ M) \in \Re^6_+ : \qquad (5)\\
&0 \leq X,\ Y,\ X_m,\ T,\ Z \leq N \leq \frac{Q}{\delta_1}, \\
&0 \leq M \leq \frac{\mu Q}{\mu_0 \delta_1} \Big\}
\end{aligned}$$

and attracts all solutions initiation in the interior of positive orthant.

2.1. Deterministic optimal control problem

In this section it is formulated and solved for deterministic version of control problem. The control variable in the model system (1), where implementation rate of awareness campaigns (μ) is represented by a Lebesgue measurable function

$u(t)$, on a finite time interval $[0,\ T_f]$. In the model $u(t)$ represents the some part of susceptible population has media awareness at time t. Our aim is to obtain optimal media awareness programs $u^*(t)$, which minimizes the number of infectives and on the other hand cost of infection (treatment) during the infectious period $[0,\ T_f]$. To investigate the optimal level of efforts that would be needed to control the disease, the objective function J is formed. Since objective of the spread of disease control is to decrease the infected individuals and infected individuals who are on treatment and increase the aware susceptible population. Hence the problem of minimizing the cost functional is,

$$J(u) = \int_0^{T_f} \left\{ AY + BT - CX_m + \frac{C_1}{2}u^2 \right\} dt \tag{6}$$

subject to

$$\begin{aligned} \frac{dX}{dt} &= Q - \beta_1 XY - \beta_2 XT - \pi XM + \pi_0 X_m \\ &\quad + \gamma_0 Z - \delta_1 X \\ \frac{dY}{dt} &= \beta_1 XY + \beta_2 XT - (\delta + \gamma + \phi + \delta_1) Y \\ \frac{dX_m}{dt} &= \pi XM - \pi_0 X_m - \delta_1 X_m \\ \frac{dT}{dt} &= \phi Y - \sigma_1 \gamma T - \delta_1 T - \sigma_2 \delta T \\ \frac{dZ}{dt} &= \gamma Y + \sigma_1 \gamma T - \gamma_0 Z - \delta_1 Z \\ \frac{dM}{dt} &= u(t)(Y + T) - \mu_0 M \end{aligned} \tag{7}$$

where, $X > 0,\ Y > 0,\ X_m \geq 0,\ T \geq 0,\ Z \geq 0$ and $M \geq 0$. A, B and C are the positive weights. The term $\frac{C_1}{2}$ is the cost associated with $u(t)$. An optimal control $u^*(t)$ is such that

$$J(u^*(t)) = \min_{u \in U} J(u(t)) \tag{8}$$

where control set is defined as

$$\begin{aligned} U = \{u(t) : 0 \leq u(t) \leq 1,\ 0 \leq t \leq T, \\ u(t) \textit{ is Lebesgue measurable}\}. \end{aligned} \tag{9}$$

2.2. Existence of deterministic optimal control problem

The existence of optimal control can be proved by using the result from Fleming and Rishel [8] .

Theorem 1. *For the optimal control problem (6) and (7) on a fixed interval* $[0,\ T_f]$*, there exist an optimal control* $u^*(t) \in U$*.*

Proof. The boundedness of solution of system (7) asserts the existence of solution to control system using results by [15], therefore, set of controls and corresponding state variables are non- empty. The control set is closed and convex by definition. The solution of system (7) are bounded above by linear function in control and state. The integrand in cost functional, $AY + BT - CX_m + \frac{C_1}{2}u^2$, is convex on control set U. Further, there exists $p,\ q > 0$ and $b > 1$ such that, $AY + BT - CX_m + \frac{C_1}{2}u^2 \geq p + q|u(t)|^b$, where p depends upon the upper bound of $Y(t),\ T(t)$ and $X_m(t)$ and $q = C_1$. Hence the existence of an optimal control is established. □

2.3. Characterization of optimal control

The Pontryagin's Maximum principle converts the problem of minimizing the cost functional subject to state variables into minimizing the Hamiltonian with respect to the controls at each time t. For the purpose of simplicity it is introduced the functions f_1, f_2, f_3, f_4, f_5 and f_6, to right side expressions of equations (7).

$$\begin{aligned} f_1(t) &= Q - \beta_1 XY - \beta_2 XT - \pi XM \\ &\quad + \pi_0 X_m + \gamma_0 Z - \delta_1 X \\ f_2(t) &= \beta_1 XY + \beta_2 XT - (\delta + \gamma + \phi + \delta_1) Y \\ f_3(t) &= \pi XM - \pi_0 X_m - \delta_1 X_m \\ f_4(t) &= \phi Y - \sigma_1 \gamma T - \delta_1 T - \sigma_2 \delta T \\ f_5(t) &= \gamma Y + \sigma_1 \gamma T - \gamma_0 Z - \delta_1 Z \\ f_6(t) &= u(t)(Y + T) - \mu_0 M \end{aligned} \tag{10}$$

Therefore Hamiltonian H is,

$$\begin{aligned} H &= AY + BT - CX_m + \frac{C_1}{2}u^2 \\ &+ \lambda_1 f_1(t) + \lambda_2 f_2(t) + \lambda_3 f_3(t) \\ &+ \lambda_4 f_4(t) + \lambda_5 f_5(t) + \lambda_6 f_6(t) \end{aligned} \tag{11}$$

where λ_i for $i = 1, 2...6$ are adjoint functions associated with their respective state variables. The necessary conditions that an optimal control problem must satisfy Hamiltonian H comes from the Pontryagins maximum principle [7]. Given an optimal control and corresponding states, there exists adjoint variable λ_i satisfying the following equations:

$$\begin{aligned}
\lambda_1' = -\frac{\partial H}{\partial X} &= \beta_1 Y(\lambda_1-\lambda_2)+\beta_2 T(\lambda_1-\lambda_2)\\
&\quad +\pi M(\lambda_1-\lambda_3)+\lambda_1\delta_1\\
\lambda_2' = -\frac{\partial H}{\partial Y} &= -A+\beta_1 X(\lambda_1-\lambda_2)+\lambda_2(\delta_+\delta_1)\\
&\quad +\phi(\lambda_2-\lambda_4)+\gamma(\lambda_2-\lambda_5)-\lambda_6\mu\\
\lambda_3' = -\frac{\partial H}{\partial X_m} &= C+\pi_0(\lambda_1+\lambda_3)+\lambda_3\delta_1\\
\lambda_4' = -\frac{\partial H}{\partial T} &= -B+\beta_2 X(\lambda_1-\lambda_2)+\lambda_4\sigma_2\delta\\
&\quad +\sigma_1\gamma(\lambda_4-\lambda_5)+\lambda_4\delta_1-\lambda_6\mu\\
\lambda_5' = -\frac{\partial H}{\partial Z} &= \lambda_5(\gamma_0+\delta_1)-\lambda_1\gamma_0\\
\lambda_6' = -\frac{\partial H}{\partial M} &= \pi X(\lambda_1-\lambda_3)+\lambda_6\mu_0 \qquad (12)
\end{aligned}$$

with transversality conditions $\lambda_i(T) = 0$, for $i = 1, 2...6$. The transversality conditions are zero because the objective functional is independent of states at the final time.

The Hamiltonian is minimized with respect to $u(t)$ at the optimal value $u^*(t)$. Since

$$\begin{aligned}
H &= AY+BT-CX_m+\frac{C_1}{2}u^2\\
&\quad +\lambda_6\{u(t)(Y+T)\}+terms\ without\ u(t),
\end{aligned}$$

differentiating H with respect to u and according to Pontrygins Maximum Principle, the unrestricted optimal control $u^*(t)$ satisfies $\frac{\partial H}{\partial u} = 0$ at $u(t) = u^*(t)$. So it is given by

$$u^*(t) = \min\left[\max\left(0, -\frac{\lambda_6(Y+T)}{C_1}\right), 1\right] \qquad (13)$$

Therefore we have the following theorem.

Theorem 2. *The optimal control $u^*(t)$ of a system (7), which minimizes the objective functional (6) is characterized by (13).*

Due to a priori boundedness of the state and adjoint system functions and the resulting Lipschitz structure of the ODE's, it is obtained the uniqueness of the optimal control for small T. The state system coupled with the adjoint system, with the initial conditions, the transversality condition together with the above characterization of the control form the optimality system.

3. Stochastic Model

In this section a non-linear stochastic SIR type epidemic model is proposed by introducing a noise in system (7), and transformed the deterministic problem into a corresponding stochastic problem. The noise can induce non-trivial effects in physical and biological systems. The presence of noise source modifies the behavior of corresponding deterministic evolution of the system to stochastic system. The real spread of infectious disease, due to variation in the environment and the weather will exhibit some kinds of random fluctuation in the infection and other variables. Here it is considered the perturbed transmission coefficients β_1 and β_2 in system (7), and hence the infection rate is replaced by

$$\beta_1 \to \beta_1 + \epsilon\eta(t) \qquad \beta_2 \to \beta_2 + \epsilon\eta(t) \quad (14)$$

where $\eta(t)$ represents the Gaussian white noise with zero mean and unit co-variance and ϵ is a constant. The relation between the Wiener process $W(t)$ and Gaussian white noise $\eta(t)$ such that $dW(t) = \eta(t)dt$, then the stochastic version of the corresponding deterministic system (7) takes the following form:

$$\begin{aligned}
dX &= [Q-\beta_1 XY-\beta_2 XT-\pi XM+\pi_0 X_m\\
&\quad +\gamma_0 Z-\delta_1 X]\,dt-\epsilon X(Y+T)dW(t)\\
dY &= [\beta_1 XY+\beta_2 XT-(\delta+\gamma+\phi+\delta_1)Y]dt\\
&\quad +\epsilon X(Y+T)dW(t)\\
dX_m &= [\pi XM-\pi_0 X_m-\delta_1 X_m]dt\\
dT &= [\phi Y-\sigma_1\gamma T-\delta_1 T-\sigma_2\delta T]dt\\
dZ &= [\gamma Y+\sigma_1\gamma T-\gamma_0 Z-\delta_1 Z]dt\\
dM &= [u(t)(Y+T)-\mu_0 M]dt \qquad (15)
\end{aligned}$$

where, $X > 0,\ Y > 0,\ X_m \geq 0,\ T \geq 0,\ Z \geq 0$ and $M \geq 0$.

In this process, it is assumed that $W(t)$ is one dimensional real Wiener process defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. For some $n \in N$, some $x_0 \in \Re^n$, and an n-dimensional Wiener process $W(t)$, consider the general n-dimensional stochastic differential equation,

$$dx(t) = F(x(t),t)dt + G(x(t),t)dW(t), \quad x(0) = x_0. \qquad (16)$$

A solution to the above equation is denoted by $x(t, x_0)$. It is assumed that $F(t, 0) = G(t, 0) = 0\ \forall\ t \geq 0$, so that the origin point is an equilibrium of (16). Let us denote by L the differential operator associated with the function displayed in

(16), defined for a function $U(t,x) \in C^{1,2}(\Re X \Re^n)$ by

$$LU = \frac{\partial U}{\partial t} + F^{trp}\frac{\partial U}{\partial x} + \frac{1}{2}Trc\left[G^{trp}\frac{\partial^2 U}{\partial x^2}G\right]. \quad (17)$$

Here trp denotes the transpose and Trc means trace of a matrix. In view of Ito's formula, if $x(t) \in \Re^d$, then $dU(x,t) = LU(x,t)dt + V_x(x,t)g(x,t)dW(t)$.

3.1. Existence and uniqueness of positive solutions

In this section, using Lyapunov analysis method (mentioned in refs. [16,17]), we show that the solution of system (15) is positive and global.

Theorem 3. *There is a unique solution $X(t),\ Y(t)\ X_m(t),\ T(t),\ Z(t),\ M(t)$ of system (15) on $t \geq 0$ for any initial value $(X(0),\ Y(0)\ X_m(0),\ T(0),\ Z(0),\ M(0)) \in \Re^6_+$ and the solution will remain in $\Re^6_+$ with probability 1, namely, $(X(t),\ Y(t)\ X_m(t),\ T(t),\ Z(t),\ M(t)) \in \Re^6_+$ for all $t \geq 0$ almost surely.*

Proof. Since the coefficient of the equation are locally Lipschitz continuous for any given initial value

$$(X(0),\ Y(0),\ X_m(0),\ H(0),\ Z(0),\ M(0)) \in \Re^6_+,$$

there is a unique local solution

$$X(t), Y(t), X_m(t), T(t), Z(t), M(t)$$

on $t \in [0, \tau_e)$, where τ_e is the explosion time (see Ref. [18]). To show that this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 \geq 0$ be sufficiently large so that $X(0), Y(0), X_m(0), T(0), Z(0)$ and $M(0)$ all lie within the interval $[1/k_0, k_0]$. For each integer $k > k_0$, define the stopping time

$$\tau_k = \inf\Big\{t \in [0,\tau_e) : \min\{X,\ Y\ , X_m\ , T\ , Z\ , M\ \} \leq \frac{1}{k} \text{ or } \max\{X\ , Y\ , X_m\ , T\ , Z\ , M\ \} \geq k\Big\},$$

where throughout this section, we set $\inf \emptyset = \infty$(as usual $\emptyset$ denotes the empty set). According to the definition, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $\{X(t),\ Y(t)\ X_m(t),\ T(t),\ Z(t),\ M(t)\} \in \Re^6_+$ a.s. for all $t \geq 0$. In other words, to complete the proof, all we need to show that $\tau_\infty = \infty$ a.s. If this statement is false, then there exist a pair of constants $\tau > 0$ and $\epsilon_1 \in (0,1)$ such that

$$P\{\tau_\infty \leq \tau\} > \epsilon_1. \quad (18)$$

Hence there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq \tau\} \geq \epsilon_1 \qquad \forall\ k \geq k_1. \quad (19)$$

For $t \leq \tau_k$, we can see, for each k,
$dN(t) = [Q - \delta(Y + \sigma_2 H) - \delta_1 N(t)]dt \leq [Q - \delta_1 N]dt$
and also

$$dM = [\mu(Y+T) - \mu_0 M]dt \leq \left[\frac{\mu Q}{\delta_1} - \mu_0 M\right]dt$$

and since

$$\frac{dM}{dt} \leq \frac{\mu Q}{\delta_1} - \mu_0 M$$

and so,

$$N(0) = X(0) + Y(0) + X_m(0) + T(0) + Z(0)$$

$$N(t) \leq \begin{cases} Q/\delta_1, & if\ N(0) \leq Q/\delta_1, \\ N(0), & if\ N(0) \geq Q/\delta_1 \end{cases} := P$$

$$M(t) \leq \begin{cases} \frac{\mu Q}{\delta_1 \mu_0}, & if\ M(0) \leq \frac{\mu Q}{\delta_1 \mu_0}, \\ M(0), & if\ M(0) \geq \frac{\mu Q}{\delta_1 \mu_0} \end{cases}$$

Define a C^2-function $V : \Re^6_+ \longrightarrow \Re^-_+$ by

$$\begin{aligned} dV &= (X - 1 - log\ X) + (Y - 1 - log\ Y) \\ &\quad +(X_m - 1 - log\ X_m) + (T - 1 - log\ T) \\ &\quad +(Z - 1 - log\ Z) + (m - 1 - log\ M) \end{aligned}$$

$$\begin{aligned} dV &= \left(1 - \frac{1}{X}\right)dx + \frac{1}{2X^2}(dx)^2 + \left(1 - \frac{1}{Y}\right) \\ &\quad + \frac{1}{2Y^2}(dy)^2 + \left(1 - \frac{1}{X_m}\right)dx_m \\ &\quad + \left(1 - \frac{1}{T}\right)dh + \left(1 - \frac{1}{Z}\right)dz \\ &\quad + \left(1 - \frac{1}{M}\right)dm \\ &= LVdt + \epsilon(Y - X)dW(t), \end{aligned} \quad (20)$$

where $LV : \Re^6_+ \to \Re_+$ is defined by

$$\begin{aligned}
LV &= \left(1-\frac{1}{X}\right)f_1(t)+\frac{1}{2}\epsilon^2(Y+T)^2\\
&+\left(1-\frac{1}{Y}\right)f_2(t)+\frac{1}{2}\epsilon^2X^2\\
&+\left(1-\frac{1}{X_m}\right)f_3(t)+\left(1-\frac{1}{T}\right)f_4(t)\\
&+\left(1-\frac{1}{Z}\right)f_5(t)+\left(1-\frac{1}{M}\right)f_6(t)\\
&= Q-\beta_1XY-\beta_2XT-\pi XM+\pi_0X_m+\gamma_0Z\\
&-\delta_1X-\frac{Q}{X}+\beta_1Y+\beta_2T+\pi M-\frac{\pi_0X_m}{X}-\frac{\gamma_0Z}{X}\\
&+\delta_1+\frac{1}{2}\epsilon^2(Y+T)^2+\beta_1XY+\beta_2XT\\
&-(\delta+\phi+\gamma+\delta_1)Y-\beta_1X\\
&-\frac{\beta_2T}{Y}-(\delta+\phi+\gamma+\delta_1)\\
&+\frac{1}{2}\epsilon^2X^2+\pi XM-\pi_0X_m-\delta_1X_m\\
&-\frac{\pi XM}{X_m}+\pi_0+\delta_1+\phi Y-\sigma_1\gamma T-\sigma_2\delta T\\
&-\delta_1T-\frac{\phi Y}{T}-\sigma_1\gamma+\sigma_2\delta+\delta_1+\gamma Y\\
&-\sigma_1\gamma T-\gamma_0Z-\delta_1Z-\frac{\gamma Y}{Z}-\frac{\sigma_1\gamma T}{Z}+\gamma_0+\delta_1\\
&+\mu(Y+T)-\mu_0M-\frac{\mu(Y+T)}{M}+\mu_0\\
&\le Q+5\delta_1+(\beta_1+\beta_2+\pi+2\mu+\delta)P+\delta\\
&+\gamma+\phi+\mu_0+\frac{5}{2}\epsilon^2P^2\\
&:= \tilde{D} \qquad (21)
\end{aligned}$$

Therefore

$$\begin{aligned}
&E\left[W\{X(\tau_k\wedge\tau),Y(\tau_k\wedge\tau),X_m(\tau_k\wedge\tau),T(\tau_k\wedge\tau),\right.\\
&\left.Z(\tau_k\wedge\tau),M(\tau_k\wedge\tau)\}\right]\\
&\le W\{X(0),Y(0),X_m(0),T(0),Z(0),M(0)\}\\
&+E\left[\int_0^{(\tau_k\wedge\tau)}dt\tilde{D}\right]\\
&\le W\{X(0),Y(0),X_m(0),T(0),Z(0),M(0)\} \qquad (22)\\
&+\tilde{D}\tau
\end{aligned}$$

Set $\Omega_k = (\tau_k \wedge \tau)$ Note that for every $\omega \in \Omega_k$, there is at least one of $X(\tau_k,\omega)$, $Y(\tau_k,\omega)$, $X_m(\tau_k,\omega)$, $T(\tau_k,\omega)$, $Z(\tau_k,\omega)$ and $M(\tau_k,\omega)$ that equals k or $1/k$ and hence $W\{X(\tau_k),\ Y(\tau_k),\ X_m(\tau_k),\ T(\tau_k),\ Z(\tau_k),\ M(\tau_k)\}$ is no less than $k-1-\log k$ or $1/k-1-\log k$ consequently.

$$\begin{aligned}
&W\{X(\tau_k),\ Y(\tau_k),\ X_m(\tau_k),\ T(\tau_k),\ Z(\tau_k),\ M(\tau_k)\}\\
&\qquad\ge k-1-\log k\wedge 1/k-1-\log k
\end{aligned}$$

It is then follows (19) and (22) that

$$\begin{aligned}
&W\{X(0),Y(0),X_m(0),T(0),Z(0),M(0)\}+\tilde{D}\tau\\
\ge &E\left[1_{\Omega_k}(\omega)W\{X(\tau_k),\ Y(\tau_k),\ X_m(\tau_k),\ T(\tau_k),\right.\\
&\left.Z(\tau_k),\ M(\tau_k)\}\right]\\
\ge &\epsilon\left[k-1-\log k\wedge 1/k-1-\log k\right]
\end{aligned}$$

$$\begin{aligned}
&W\{X(0),Y(0),X_m(0),T(0),Z(0),M(0)\}+\tilde{D}\tau\\
\ge &E\left[1_{\Omega_k}(\omega)W\{X(\tau_k),\ Y(\tau_k),\ X_m(\tau_k),\ T(\tau_k),\right.\\
&\left.Z(\tau_k),\ M(\tau_k)\}\right]\\
\ge &\epsilon\left[k-1-\log k\wedge 1/k-1-\log k\right]
\end{aligned}$$

where $1_{\Omega_k}(\omega)$ is the indicator function of Ω_k. Let $k \to \infty$ leads to the contradiction $\infty > W\{X(0),Y(0),X_m(0),T(0),Z(0),M(0)\} + \tilde{D}\tau = \infty$. So we must therefore have τ_∞ and hence the proof. □

Remark 1. *From theorem 3 for any initial value* $(X(0),\ Y(0)\ X_m(0),\ T(0),\ Z(0),\ M(0)) \in \Re^6$, *there is a unique global solution* $X(t),\ Y(t)\ X_m(t),\ T(t),\ Z(t),\ M(t) \in \Re^6$ *almost surely of system (15). Hence* $dN(t) \le [Q-\delta_1N(t)]dt$, *and* $N(t) \le \frac{Q}{\delta_1} + e^{-\delta_1 t}(N(0))$ *If* $N(0) \le \frac{Q}{\delta_1}$, *then* $N(t) \le \frac{Q}{\delta_1}$ *a.s. so the region*

$$\begin{aligned}
\Omega^* = &\{(X,Y,X_m,H,Z,M): X>0,\ Y>0\ X_m>0\\
&T>0,\ Z>0\ M>0, N(t)\le\frac{Q}{\delta_1}a.s.\}
\end{aligned}$$

is a positively invariant set of system (15) on Ω^*, *which is similar to* Ω *of system (1). From now on, we always assume that* $(X(0),\ Y(0)\ X_m(0),\ T(0),\ Z(0),\ M(0)) \in \Omega^*$.

3.2. Stochastic optimal control problem

In this section stochastic version of the optimal control problem (1) is formulated and discussed. For stochastic control theory refer [19] of Oksendal. Here the objective is to find an optimal media awareness programs $u^*(t)$ which minimizes the objective functional with an initial state x_0 is defined by

$$E_{0,x_0}\left[\int_0^{T_f}\left\{AY+BT-CX_m+\frac{C_1}{2}u^2\right\}ds\right] \qquad (23)$$

Here the expectation is obtained on the initial condition of the state (at time $t=0$) system is x_0. For the deterministic problem of earlier, it is assumed that there is a fixed constant $u(t) \le 1$ with $u(t) \le \overline{u}$ (a.s.). The class of admissible control laws is

$$\mathcal{A} = \{u(.) : u \quad is \quad adapted, \quad and \quad 0 \le u \le \overline{u} \quad a.s.\}. \tag{24}$$

To solve this stochastic control problem, the performance criterion is defined as follows:

$$J(t,x;u) = E_{t,x}\left[\int_t^{T_f}\left\{AY + BT - CX_m + \tfrac{C_1}{2}u^2\right\}ds\right], \tag{25}$$

where the expectation is conditional on the state of the system being a fixed value x at time t. The value function is define as

$$U(t,x) = \inf_{u(.)\in\mathcal{A}} J(t,x;u) = J(t,x;u^*). \tag{26}$$

It is determined a control law that minimizes the expected value $J : \mathcal{A} \to \Re_+$ given by (26). Now the stochastic analogue of the optimal control problem is formulated, subsequent to which the solution formulae is presented.

Problem: Given the system (16) and given $\mathcal{A}$ as in (24) with J as in (25), find the value function

$$U(t,x) = \inf_{u\in\mathcal{A}} J(t,x;u), \tag{27}$$

and an optimal control function

$$u^*(t) = arg \inf_{u\in\mathcal{A}} J(x;u(t)) \in \mathcal{A}. \tag{28}$$

An expression for the optimal media awareness program $u^*(t)$ is computed through the following theorem.

Theorem 4. *A solution to the optimal media awareness program problem stated in problem (24) is of the form*

$$u^*(t) = min\left[max\left(0, \left[\frac{-U_M(t)(Y+T)}{C_1}\right]\right), \overline{u}\right]. \tag{29}$$

Proof. To determine $u^*(t)$ through the dynamic programming approach it is necessary to calculate $LU(t)$ i.e. by using (17):

$$\begin{aligned} LU(t) = {} & f_1(t)U_X(t) + f_2(t)U_Y(t) + f_3(t)UX_m(t) \\ & + f_4(t)U_T(t) + f_5(t)U_Z(t) + f_6(t)U_M(t) \\ & + \frac{1}{2}(\epsilon X(Y+T))^2 U_{XX}(t) \\ & + \frac{1}{2}(\epsilon X(Y+T))^2 U_{YY}(t) \\ & - \frac{1}{2}(\epsilon X(Y+T))^2 U_{XY}(t). \end{aligned} \tag{30}$$

Applying the Hamilton-Jacobi-Bellman theory (see, for instance, [19])

$$\inf_{u\in\mathcal{A}}\left[AY + BT - CX_m + \frac{C_1}{2}u^2 + LU(t)\right]. \tag{31}$$

To compute the equation (31) it requires to derive partial derivative of the below given expression with respect to u, and equating to zero.

$$AY + BT - CX_m + \frac{C_1}{2}u^2 + LU(t). \tag{32}$$

This leads to the equation:

$$\begin{aligned} & C_1u(t) + U_M(t)(Y+T) = 0 \\ u^*(t) \quad = \quad & \min\left[\max\left(0, \frac{-U_M(t)(Y+T)}{C_1}\right), \overline{u}\right] \end{aligned} \tag{33}$$

□

In the following section numerical analysis of the results are discussed.

4. Numerical Simulations

The feasibility of analysis regarding deterministic optimality and stochastic optimality conditions are simulated numerically over $t = 30$ units of time. All parameter values in the computations are the same in both scenarios. The common parameter values used in the computations are $Q = 2,\ \beta_1 = 0.000007,\ \beta_2 = 0.000000006,\ \pi = 0.0000025,\ \gamma = 0.15,\ \gamma_0 = 0.0002,\ \phi = 0.005,\ \delta = 0.0001,\ \delta_1 = 0.00005,\ \mu_0 = 0.5,\ \sigma_1 = 2,\ \sigma_2 = 0.5,\ \pi_0 = 0.02,\ A_1 = 200,\ B = 250,\ C = 1,\ C_1 = 230,\ \epsilon = 0.0002$, while the initial conditions are $X = 1,00,000,\ Y = 200,\ X_m = 0,\ T = 0,\ Z = 0,\ M = 0$.

An iterative scheme of fourth order Runge-Kutta method is used for solving the deterministic optimality system. This method of numerically integrated ordinary differential equations by using trial step at midpoint of an interval to eliminate lower order errors terms. The algorithm is the forward-backward scheme; starting with an initial guess for the optimal controls, the state variables are then solved forward in time from the dynamics of system (1) using a Runge-Kutta method of the fourth order. Then, those state variables and initial guess for the controls are used to solve the adjoint equation (12) backward in time with given final conditions (13), again employing a fourth order Runge-Kutta method. The controls are updated and used to solve the state and then the adjoint system. This iterative process terminates when current state, adjoint, and control values converge sufficiently (See, [4,5]).

Numerical simulation to the system comprising state system (15) compelled with the proxy adjoint system (12) with transverslity conditions and characterization of the control variable $u^*(t)$ in equation (29) are carried out using forward backward algorithm. The state system (15), i.e., stochastic differential equations were first simulated using forth order Range-Kutta method by introducing noise through Euler Maruyama method [20] and then adjoint system (12) are simulated backward in time with final conditions(See, Witbooi et al. [14]). In particular, we use as a proxy for $\lambda_6(Y+T)$ in the calculation of $u(t)$ in this case. We note that the presence of $Y(t)$ makes $u(t)$ into a stochastic variable even with the said proxy (in the stochastic case).

Figure 1 shows the time series plot to illustrates the variation of the number of individuals in each compartment of the population and number of awareness campaigns with respect to time (in weeks) and Figure 1(a) shows time series plot for the deterministic epidemic model under the time dependent control $u(t)$ where as Figure 1(b) representing the control profile of the same model. Further it is evident from the Figure 1(b) that it is optimal to run awareness campaigns up to 29 units of time at maximum rate and lower down afterwords.

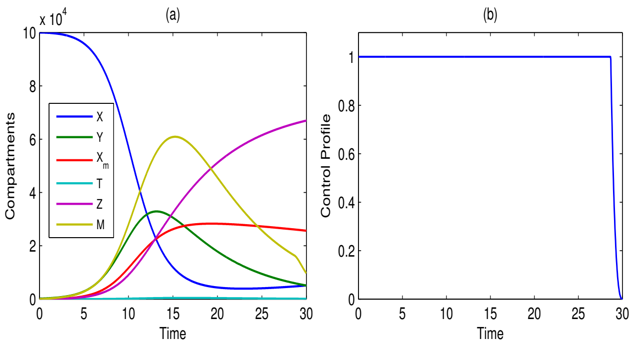

Figure 1. Simulation of deterministic model solution (a) and control profile $u(t)(b)$.

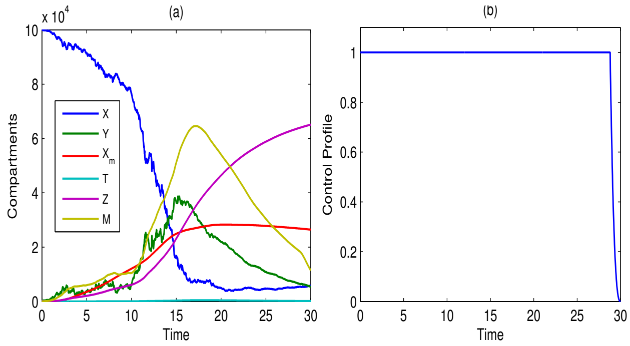

Figure 2. Simulation of Stochastic model solution (a) and control profile $u(t)(b)$.

Figure 2 illustrates the stochastic model solutions using the same parameter values and initial conditions as that of deterministic model parameters used in the illustration of Figure 1 and the corresponding control profile $u(t)$ for stochastic model. It is observed that stochastic model solution also depicts same scenario as that of deterministic model solution under the time dependent control $u(t)$ and also control profile $u(t)$ exhibits same state of affairs as that of deterministic control profile. An important point to note about our approximation is that it fully accommodates the stochasticity (embodied and concentrated in the factor Y of the expression $u^*(t)$).

To investigate how optimal control depends upon different parameters of the deterministic and stochastic model, control profile is plotted for different values of transmission rate β_1 and recovery rate γ in Figures 3 and 4 respectively. It is observed from Figure 3(a) that for higher value of transmission rate β_1, to achieve the optimal scenario awareness campaign must be implemented with maximum rate up to to 28 units of time.

However for lower value of β_1 *i.e.*, $\beta_1 = 0.000001$, and $\beta_1 = 0.0000001$, the optimal scenario can be obtained by implementing awareness campaigns with maximum rate only for initial 21 and 14 units of time, respectively. The stochastic control profile Figure 3(b) also depicts similar state of affairs, but the optimal scenario can be obtained by implementing awareness campaigns with maximum rate only for initial 7 and 8 units of time, for $\beta_1 = 0.000001$, and $\beta_1 = 0.0000001$ respectively, then onwards implementation of awareness campaigns will be guided by stochastic control profile.

These course of remedies are observed for the reason that when the transmission of disease is slow, less people get affected and hence less awareness campaigns are needed to control the disease.

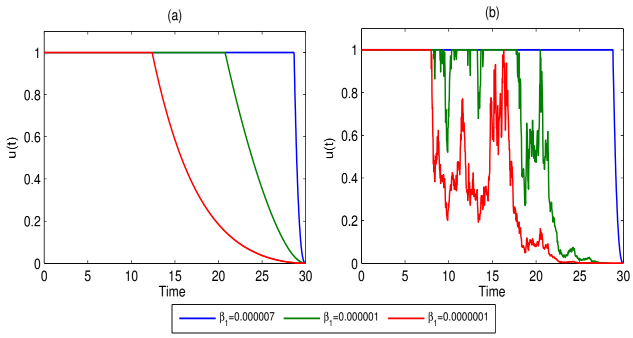

Figure 3. Simulation of Deterministic (a) and Stochastic (b) control profile for different values of β_1.

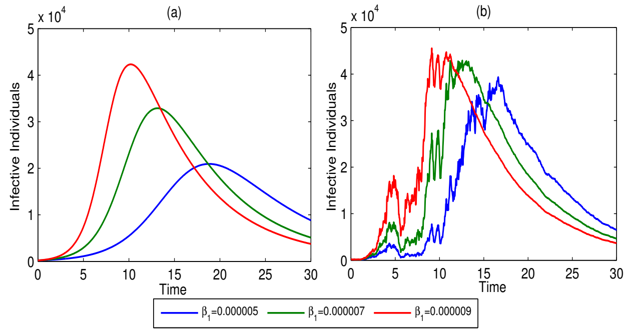

Figure 5. Simulation of Deterministic (a) and Stochastic (b) Infectives for different values of β_1.

Similarly optimal scenario will change from implementing awareness campaigns with maximum rate up to 28 units of time to 16 and 6 units of time for the change in recovery rate $\gamma = 0.15$ to $\gamma = 0.8$ and $\gamma = 1$ respectively (Figure 4(a)). Again stochastic control profile Figure 4(b) also depicts similar state of affairs, but the optimal scenario can be obtained by implementing awareness campaigns with maximum rate only for initial 6 and 7 units of time, for $\gamma = 0.8$ and $\gamma = 1$ respectively, then onwards implementation of awareness campaigns will be guided by stochastic control profile.

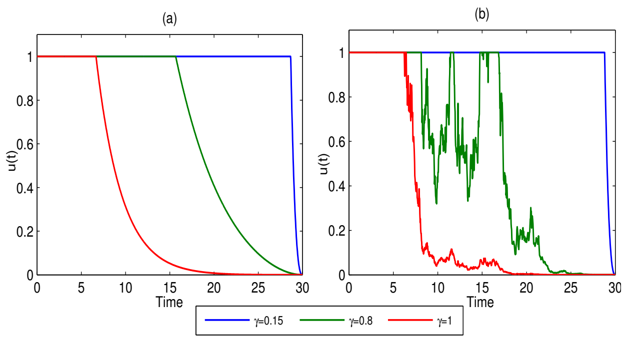

Figure 4. Simulation of Deterministic (a) and Stochastic (b) control profile for different values of recovery rate γ.

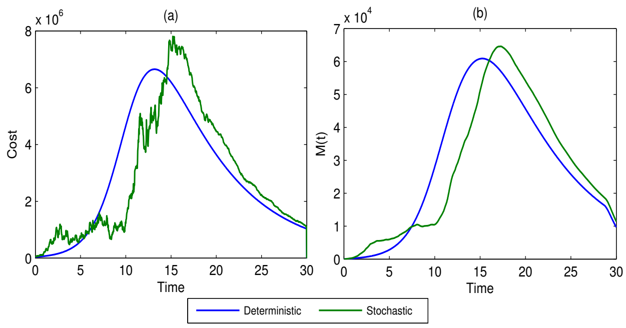

Figure 6. Simulation of Deterministic and Stochastic Cost (a) and Media campaigns (b).

Figure 5 shows the effect of transmission rate β_1 on infected population for the deterministic and stochastic models. Increase in the transmission rate β_1 leads increase in number of infections, and hence it requires to continue the implementation of awareness campaigns at maximum rate.

Figure 6(a) shows the simulation of deterministic and stochastic cost function and cumulative density of awareness programs 6(b). From the Figures 6(a) and 6(b) it is clear that cost and awareness programs are proportional to each other, which implies that, as number of media awareness programs increases cost of control for epidemic is also increasing.

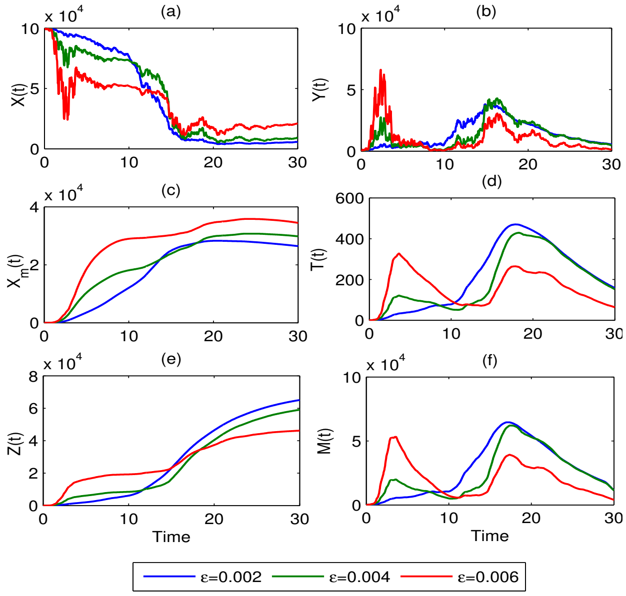

Figure 7. Simulation of solution of each states for different values of perturbation parameter ϵ.

Figure 7 shows the difference in the number of individuals in each state of a system (15) for different values of perturbation parameter ϵ. From Figure 7(a) it is observed that the number of unaware susceptible are decreasing as ϵ increases initially up to 12 units of time and then increasing till final time. In case of infectives as perturbation increases number of infections are increasing up to 8 units of time later it is decreasing till final time see Figure 7(b) and from Figure 7(c) it is clear that as perturbation increases, aware susceptible are increasing till final time. Figure 7(d),7(e),7(f) are varying in the same direction as that of infectives, as ϵ increases.

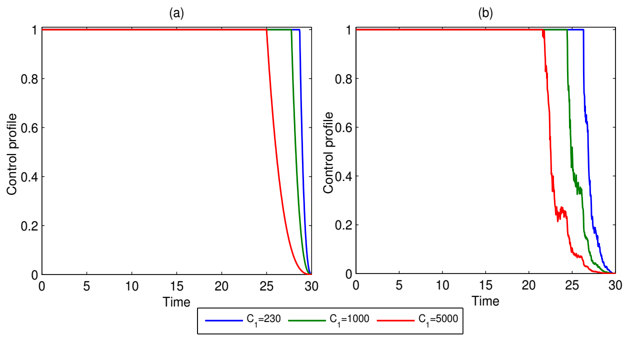

Figure 8. Simulation of control profile for different values of C_1.

To investigate how the optimal control varies depends upon the positive weight C_1, it is plotted the control profile for different values of C_1. It is observed from the Figure 8(a) that as the positive weight C_1 increased up to 5000 the optimal scenario is achieved in 25 units of time and when $C_1 = 230$ it is sufficient to implement control on awareness programs at maximum rate up to 29 units of time. For stochastic optimal control it is observed from Figure 8(b) that when $C_1 = 5000$ it is enough to implement optimality at maximum rate up to 22 units of time and for lower value of $C_1 = 230$ it is necessary to continues the implementation of awareness programs up to 27 units of time at maximum rate. This indicates that as the weight of control (awareness programs) increases, the disease can be controlled in a minimum time.

5. Conclusion

Media campaigns and epidemics are closely related to each other. The bases of this association is human behavioral responses. The present study considered the optimal control analysis of both deterministic differential equation modeling and stochastic differential equation modeling of infectious disease by taking effects of media awareness programs and treatment of infectives on the epidemic into account. Optimal media awareness strategy under the quadratic cost functional using Pontrygin's Maximum Principle and Hamiltonian-Jacobi-Bellman equation are derived for both deterministic and stochastic optimal control problem respectively. The Hamiltonian-Jacobi-Bellman equation is used to solve stochastic system, which is fully non-linear equation, however it ought to be pointed out that for stochastic optimality system, it may be difficult to obtain the numerical results. For the analysis of the stochastic optimality system, the results of deterministic control problem are used to find an approximate numerical solution for the stochastic control problem. Outputs of the simulations shows that media awareness programs place important role in the minimization of infectious population with minimum cost. The model analysis further shows that awareness programs through the media campaigning are helpful in decreasing the spread of infectious diseases by isolating a fraction of susceptible from infectives. Numerical simulation of stochastic optimal control problem enables to measure the feasibility of option followed. A formal approach to the numerical simulation of the stochastic optimal control problem is far more complex and labour intensive and our method is a workable approximate alternative.

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RESEARCH ARTICLE

An integral formulation for the global error of Lie Trotter splitting scheme

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ABSTRACT

An ordinary differential equation (ODE) can be split into simpler sub equations and each of the sub equations is solved subsequently by a numerical method. Such a procedure involves splitting error and numerical error caused by the time stepping methods applied to sub equations. The aim of the paper is to present an integral formula for the global error expansion of a splitting procedure combined with any numerical ODE solver.

1. Introduction

Consider an autonomous ODE system in a real Banach space

$$\frac{du}{dt} = (A+B)u, \qquad u(0) = u_0, \tag{1}$$

where A and B are Lie operators allowing us to write the formal solution as

$$\begin{aligned} u(t) = \varphi_t^{A+B} u_0 &= e^{t(A+B)} u_0 \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}(A+B)^k u_0, \end{aligned} \tag{2}$$

The solutions of sub equations

$$\frac{du}{dt} = Au \qquad \text{and} \qquad \frac{du}{dt} = Bu, \tag{3}$$

can be merged within a small time step h by

$$u_{n+1} = e^{hb_1 B} e^{ha_1 A} e^{hb_2 B} \dots e^{ha_m A} e^{hb_{m+1} B} u_n,$$

or equivalently,

$$u_{n+1} = (\varphi^B_{hb_{m+1}} \circ \varphi^A_{ha_m} \circ \dots \varphi^B_{hb_2} \circ \varphi^A_{ha_1} \circ \varphi^B_{hb_1}) u_n,$$

where u_n and u_{n+1} are approximations at $t = t_n$ and $t = t_{n+1}$ with $h = t_{n+1} - t_n$. The reverse orders of A and B as well as a_i and b_i should be noticed. This happens when one applies Lie transforms to their corresponding maps. This phenomena is termed as *Vertauschungssatz* in the literature [1]. One of the sub problems in (3) (or both) can be solved numerically. When a splitting procedure and a numerical solver are of p^{th} and r^{th} order respectively, we are interested in the integral form of the leading term of the global error.

Although it is very classical subject of numerical analysis, the global error analysis of the numerical solvers for ODEs has been discussed by Viswanath [2] and Iserles [3] in different aspects. Viswanath employed Lyapunov's exponents to express error patterns of numerical solvers for hyperbolic problems. However, Iserles presented a way of deriving an asymptotic formula for the global error in the numerical solution of highly oscillatory problems.

Error bound for splitting schemes is an active research area. The splitting of bounded operators was analyzed in [1, 4]. Jahnke and Lubich [5] found error bounds for the Strang splitting in the presence of unbounded operators, which corresponds to splitting a time dependent PDE without discretization of space operators. Hansen and Ostermann [6] also presented error analysis of splitting schemes for unbounded operators in the content of semigroup theory. Apart from the above mentioned approaches, Csomos and Farago [7] discussed the interaction of the error caused by numerical methods employed for sub problems and splitting schemes. Our main task is to give clear integral representation of this interaction. In this work, we propose to approximate the global error in terms of the local errors and the discrete flow by a Riemann integral.

2. Preliminaries

We would like to explain some of notations which will be used in the later sections. Consider the initial value problem

$$\mathbf{y}' = f(t, \mathbf{y}), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{4}$$

where $\mathbf{y}_0 \in \mathbb{R}^m$ and $f : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous. When a small perturbation is introduced to the initial value $\mathbf{y}_0$, for the perturbed solution $\tilde{\mathbf{y}}(t)$, the error $\mathbf{e}(t) = \mathbf{y}(t) - \tilde{\mathbf{y}}(t)$, evolves with [3,8]

$$e(t) = \Psi(t)\Psi^{-1}(c)e(c) + \mathcal{O}(e(c)^2), t > c > 0, \tag{5}$$

where $\Psi(t)$ satisfies the variational equation

$$\Psi'(t) = J(t)\Psi(t), \qquad \Psi(0) = I, \tag{6}$$

where $J(t) = \dfrac{\partial f}{\partial \mathbf{y}}$. In order to use exponentials in defining flows we firstly express (4) an autonomous system as

$$\frac{dt_1}{dt} = 1, \tag{7}$$

$$\frac{d\mathbf{y}}{dt} = f(t_1, \mathbf{y}), \tag{8}$$

and then define a Lie operator as follows

$$L = \frac{\partial}{\partial t_1} + f(t_1, \mathbf{y})\frac{\partial}{\partial \mathbf{y}}, \tag{9}$$

which enables us to express (4) as

$$\frac{du}{dt} = Lu, \qquad u(0) = u_0, \tag{10}$$

where $u = (t_1, y)^T$ and the formal solution is $u(t) = \varphi_t^L(u_0) = e^{tL}u_0$.

3. Motivation

The local error ($\mathfrak{le}$) of a numerical method $u_{n+1} = R_{\Delta t}(u_n)$ with step size Δt for the initial value problem

$$\frac{du}{dt} = Lu, \qquad u(t_i) = u_i, \tag{11}$$

is given by

$$\Delta t^{r+1}\mathfrak{le}(u_n) = R_{\Delta t}(u_n) - \varphi_{\Delta t}^L(u_n) + \mathcal{O}(\Delta t^{r+2}). \tag{12}$$

The global error is defined as

$$\begin{aligned} e_{n+1} &= u_{n+1} - u(t_{n+1}), \\ &= R_{\Delta t}(u_n) - \varphi_{\Delta t}^L(u(t_n)). \end{aligned}$$

Therefore

$$\begin{aligned} e_{n+1} =& \Delta t^{r+1}\mathfrak{le}(u_n) + \varphi_{\Delta t}^L(u_n) - \varphi_{\Delta t}^L(u(t_n)) \\ &+ \mathcal{O}(\Delta t^{r+2}). \end{aligned} \tag{13}$$

The difference $\varphi_{\Delta t}^L(u_n) - \varphi_{\Delta t}^L(u(t_n))$ can be interpreted as the time evolution of a small perturbation to initial condition $u(t_n)$ within a time interval of which length is Δt. As a result of this interpretation and considering (5), one obtains

$$\begin{aligned} \varphi_{\Delta t}^L(u_n) - \varphi_{\Delta t}^L(u(t_n)) =& \Psi(t_{n+1})\Psi^{-1}(t_n)e_n \\ &+ \mathcal{O}(\|e_n^2\|), \end{aligned} \tag{14}$$

where $\Psi(t)$ is the solution of variational equation of the corresponding initial value problem. Therefore the first order difference equation for global error is given by

$$\begin{aligned} e_{n+1} =& e_n\Psi(t_{n+1})\Psi^{-1}(t_n) + \Delta t^{r+1}\mathfrak{le}(u(t_n)) \\ &+ \mathcal{O}(\Delta t^{r+2}). \end{aligned} \tag{15}$$

A careful reader notices that $u(t_n)$ is substituted in the term $\mathfrak{le}$ instead of u_n. It might be assumed that the difference is included in the term $\mathcal{O}(\Delta t^{r+2})$ as Iserles pointed out in [3].

Assuming $e_i = 0$, the solution of the linear difference equation is

$$e_n = \Delta t^{r+1}\Psi(t_n)\sum_{k=i}^{f-1}\Psi^{-1}(t_{k+1})(\mathfrak{le}(u(t_k)) + \mathcal{O}(\Delta t^{r+2})). \tag{16}$$

For $t_f - t_i = h = m\Delta t$, the error can be written in the integral form

$$e(t_f) = \Delta t^r\Psi(t_f)\int_{t_i}^{t_f}\Psi^{-1}(\tau+\Delta t)\mathfrak{le}(u(\tau))d\tau + \mathcal{O}(\Delta t^{r+1}).$$

As an example, we derive the global error of Euler method for the linear problem

$$\frac{du}{dt} = -\frac{1}{t+1}u(t), \qquad u(t_i) = u_i. \tag{17}$$

We will find an estimation for the actual error at $t_f = t_i + h$ with time step $\Delta t = \frac{h}{m}$. It is known that local error coefficient for Euler method (in terms of Lie Operator)

$$\mathfrak{le}(u(t)) = -1/2L^2(u(t)) = -\frac{u(t)}{(t+1)^2}. \tag{18}$$

The variational flow is determined by solving

$$\frac{d\Psi}{dt} = J(t)\Psi, \qquad \Psi(t_i) = 1, \tag{19}$$

where $J(t) = -\dfrac{1}{t+1}$. Therefore,

$$e(t_f) = \Delta t\Psi(t_f)\int_{t_i}^{t_f}\Psi(\tau+\Delta t)^{-1}\mathfrak{le}(u(t))d\tau + \mathcal{O}(\Delta t^2), \tag{20}$$

$$e(t_f) = \Delta t\frac{t_i+1}{t_f+1}\int_{t_i}^{t_f}\frac{\tau+1+\Delta t}{t_i+1}\left(\frac{-u(\tau)}{(\tau+1)^2}\right)d\tau + \mathcal{O}(\Delta t^2), \tag{21}$$

where $u(\tau) = u_i\dfrac{t_i+1}{\tau+1}$. Finally, one obtains the formula

$$e(t_f) \approx u_i\Delta t\left(\frac{1+t_i}{1+t_f}\right)\left(-1/2\,\frac{(\Delta t+2+2\,t_i)}{(1+t_i)^2} + 1/2\,\frac{(\Delta t+2+2\,t_f)}{(1+t_f)^2}\right), \tag{22}$$

that predicts the global error at $t = t_f$ in terms of initial value u_i at $t = t_i$ and step size Δt.

4. Global error of Lie Trotter Splitting

In this section, the above mentioned procedure is modified to obtain the global error expansion of any splitting procedure combined with any ODE solver. For clarity, the derivation of the formulas are given for Lie-Trotter that is widely used in the literature. The extension to the higher splitting schemes can be done in a similar way. Another simplification is that one part is assumed to be solved exactly and the other part is solved numerically.

Consider the scheme

$$u_{n+1} = [R^A_{\Delta t}]^{(m)}(\varphi^B_h(u_n)), \tag{23}$$

indicating that the sub equation $u' = Bu$ is solved exactly in $[t_n, t_{n+1}]$ and the sub equation $u' = Au$ is solved by r^{th} order numerical method $R^A_{\Delta t}$ $(r > 1)$in $[t_n, t_{n+1}]$ with step size $\Delta t = \frac{h}{m}$ (m step in each sub interval). Such a procedure involves the following two local errors

$$\begin{aligned}\Delta t^r\mathfrak{le}_{\mathfrak{R}}(\varphi^B_h(u_n)) =& [R^A_{\Delta t}]^{(m)}(\varphi^B_h(u_n)) - \varphi^{A+B}_h(u_n) + \mathcal{O}(\Delta t^{r+1}),\\ h^2\mathfrak{le}_{\mathfrak{S}}(u_n) =& \varphi^A_h\circ\varphi^B_h(u_n) - \varphi^{A+B}_h(u_n) + \mathcal{O}(h^3),\end{aligned} \tag{24}$$

where $\mathfrak{le}_{\mathfrak{S}}(u_n) = \frac{1}{2}[B,A]$ is the coefficient of the leading term of the local splitting error (Lie Trotter in this case). $\mathfrak{le}_{\mathfrak{R}}(\varphi^B_h(u_n))$ should be considered as the global error of $R^A_{\Delta t}$ at $t_f = t_{n+1}$ starting from $t_i = t_n$ with step size Δt. This kind of global error terms of ODE solvers can be computed by method described in the motivation section. (See 20 in case of Euler method). The term $\mathfrak{le}_{\mathfrak{R}}(\varphi^B_h(u_n))$ also warns us to compute the error of the method $R^A_{\Delta t}$ at the point $\varphi^B_h(u_n)$ not at the point u_n. This is the key issue in the derivation error formulas for the splitting schemes.

Consider the partition $0 = t_0 < t_1 < t_2 < ... < T$ of the interval $[0, T]$. The global error is defined by

$$e_{n+1} = u_{n+1} - u(t_{n+1}) = [R^A_{\Delta t}]^{(m)} \circ \varphi^B_h(u_n) - \varphi^{(A+B)}_h(u(t_n)).$$

Adding and subtracting the terms $(\varphi^A_h \circ \varphi^B_h)(u_n)$ and $\varphi^{(A+B)}_h(u_n)$ yields

$$\begin{aligned} e_{n+1} =&[R^A_{\Delta t}]^{(m)} \circ \varphi^B_h(u_n) - (\varphi^A_h \circ \varphi^B_h)(u_n) \\ &+ (\varphi^A_h \circ \varphi^B_h)(u_n) - \varphi^{(A+B)}_h(u_n) \\ &+ \varphi^{(A+B)}_h(u_n) - \varphi^{(A+B)}_h(u(t_n)). \end{aligned}$$

Grouping the terms two by two and considering (24) and (14) one can write

$$\begin{aligned} e_{n+1} =&\Delta t^r \mathfrak{le}_{\mathfrak{R}}(\varphi^B_h(u_n)) + h^2 \mathfrak{le}_{\mathfrak{S}}(u_n) \\ &+ e_n \Psi(t_{n+1})\Psi^{-1}(t_n) + \mathcal{O}(h^3) + \mathcal{O}(\Delta t^{r+1}), \end{aligned} \tag{25}$$

where Ψ is the solution of variational equation that corresponds to the full equation (1).

After approximating the solution of this difference equation as a Riemann integral, the global error in the integral form is computed by

$$\begin{aligned} e(T) =&h\Psi(T) \int_0^T \Psi^{-1}(t+h)\{\frac{\Delta t^r}{h} \mathfrak{le}_{\mathfrak{R}}\left(\varphi^B_h\left(u(t)\right)\right) \\ &+ \frac{1}{2}[B,A]\}dt + \mathcal{O}(h^2) + \mathcal{O}(\Delta t^{r+1}h^{-1}). \end{aligned} \tag{26}$$

5. Numerical Example

In this section, we will show the sharpness of the estimation of the global errors given by (26). As a test equation we choose

$$\frac{du}{dt} = -\frac{u(t)}{t+1} - u^2(t), \qquad u(0) = 1, \tag{27}$$

with exact solution

$$u(t) = \frac{1}{(ln(t+1)+1)(t+1)},$$

The sub equations

$$\frac{du}{dt} = -\frac{u(t)}{t+1}, \qquad u(0) = u_0,$$

and

$$\frac{du}{dt} = -u^2(t), \qquad u(0) = u_0,$$

have the exact solutions $u_A(t) = \dfrac{u_0}{t+1}$ and $u_B(t) = \dfrac{u_0}{1+tu_0}$, respectively. One also needs the variational flows of the equations (27) which can be given as

$$\Psi_{full}(t) = \frac{1}{(ln(t+1)+1)^2(t+1)}. \tag{28}$$

When the part A is solved by first order Euler method with step size $\Delta t = \frac{h}{m}$ in $[t_n, t_{n+1}]$ and part B proceeds in time by its exact flow, the numerical scheme is written as

$$u_{n+1} = [R^A_{\Delta t}]^{(m)} \circ \varphi^B_h(u_n). \tag{29}$$

Firstly the term $\mathfrak{le}_{\mathfrak{R}}\left(\varphi^B_h(u(t))\right)$ that is, the global error of Euler time stepping at $t+h$ starting from t with initial condition $\varphi^B_h(u(t))$ is needed. Luckily, the desired error formula, but with initial condition u_i, has been already derived in (22). Just only taking $t_i = t$, $t_f = t+h$ and $u_i = \varphi^B_h(u(t)) = \dfrac{u(t)}{1+hu(t)}$, one should see

$$\begin{aligned} \Delta t \mathfrak{le}_{\mathfrak{R}}(\varphi^B_h(u(t))) =& [R^A_{\Delta t}]^{(m)}(\varphi^B_h(u(t))) \\ &-(\varphi^A_h \circ \varphi^B_h)(u(t)), \\ =& \frac{u(t)(1+t)}{2(1+hu(t))(1+t+h)}\left(\frac{(\Delta t+2+2\tau)}{-(1+t)^2} + \frac{(\Delta t+2+2t+h)}{(1+t+h)^2}\right). \end{aligned}$$

On the other hand, the leading coefficient of Lie Trotter splitting for (27) is found to be

$$\mathfrak{le}_{\mathfrak{s}}(u(t)) = \frac{1}{2}[B,A]u(t) = -\frac{1}{2}\frac{u(t)^2}{t+1}. \tag{30}$$

Finally, computing the integral (26) yields the estimation

$$\begin{aligned} e(T) \approx& h\Psi_{full}(T) \int_0^T \Psi^{-1}_{full}(\tau+h) \\ &\left\{\frac{1}{m}\mathfrak{le}_{\mathfrak{R}}(\varphi^B_h(u(\tau))) + \mathfrak{le}_{\mathfrak{S}}(u(\tau))\right\} d\tau. \end{aligned} \tag{31}$$

Table I presents the sharpness of the estimation (31) for various Δt and h at final $T = 20$.

Table 1. Comparison of actual errors and estimated errors of Lie Trotter at $T = 20$.

	$h = 0.1$ Δt=0.01	$h = 0.1$ Δt=0.001	$h = 0.01$ Δt=0.001	$h = 0.01$ Δt=0.0001
Actual error	-1.909e-4	-1.445e-4	-1.899e-5	-1.438e-5
Estimated error	-1.621e-4	-1.575e-4	-1.409e-5	-1.404e-5

6. Remarks and Conclusion

Splitting methods are becoming more and more popular among practitioners of numerical methods for differential equations. These methods provide separate treatments of simpler sub equations comparing to whole problem. However, the interaction of the errors caused by splitting procedure and time stepping methods applied to sub problems should be considered because the interaction might lead to order reduction in the long time run. Such a derived formula enables us to estimate error behavior of a method so that suitable solvers are employed. We choose a simple test problem to give a clear description of the integral formula. However in most of the applied problems, exact flows of full equation and sub equations are not available. In this case, derived formulas can be used to obtain reasonable error bounds by taking appropriate norms of the given expressions. However, in case of long time integration, asymptotic solutions and asymptotics expansions of the corresponding integrals that can be computed by some perturbation methods such as WKB give the long time error behaviors of the numerical methods. Indeed, the presented formulas are derived in search of suitable splitting algorithms for the long time integration of highly oscillatory non linear equations.

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RESEARCH ARTICLE

On refinements of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral operators

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ABSTRACT

In this paper, we first establish weighted versions of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral operators utilizing weighted function. Then we obtain some refinements of these inequalities. The results obtained in this study would provide generalization of inequalities proved in earlier works.

1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [17, p.137], [2]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave.

In [6], Fejér obtained the following inequality which is the weighted generalization of Hermite-Hadamard inequality (1):

Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x) &\leq \int_a^b f(x)g(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \end{aligned}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a + b)/2$.

A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend these two inequalities for different classes of functions, (see, for example, [1]- [5], [8]- [11], [13], [14], [16], [19]- [26]) and the references cited therein.

The remainder of this work is organized as follows: we first give the definitions of Riemann-Liouville fractional integrals and present some Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral operators in Section 2. In the main section, we first establish a new weighted version of Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. Moreover, we obtain some refinements of this result using the symmetric weighted function. We give also some special cases of these inequalities. In the last section, we give some conclusions and future directions of research.

2. Preliminaries

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

It is remarkable that Sarikaya et al. [20] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \qquad (2) \\ \leq\ & \frac{f(a)+f(b)}{2} \end{aligned}$$

with $\alpha > 0$.

Hermite-Hadamard-Fejér inequality for Riemann-Liouville fractional integral operators was given by İşcan in [11], as follows:

Let $f : [a,b] \to \mathbb{R}$ be convex function with with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{a+b}{2}$ i.e. $g(a+b-x) = g(x)$, then the following inequalities hold

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}(g)(b) + J_{b-}^{\alpha}(g)(a)\right] \\ \leq\ & \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right] \\ \leq\ & \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}(g)(b) + J_{b-}^{\alpha}(g)(a)\right]. \end{aligned}$$

For more information for fractional calculus, please refer to ([7], [12], [15], [18]).

Now we give the following lemma:

Lemma 1. *[22,25] Let $f : [a,b] \to \mathbb{R}$ be a convex function and h be defined by*

$$h(t) = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{t}{2}\right) + f\left(\frac{a+b}{2} + \frac{t}{2}\right)\right].$$

Then h is convex, increasing on $[0, b-a]$ and for all $t \in [0, b-a]$,

$$f\left(\frac{a+b}{2}\right) \leq h(t) \leq \frac{f(a)+f(b)}{2}.$$

In [22], Xiang obtained following important inequalities for the Riemann-Liouville fractional integrals utilizing the Lemma 1:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then WH is convex and monotonically increasing on $[0,1]$ and*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= WH(0) \leq WH(t) \leq WH(1) \qquad (3) \\ &= \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[\left(J_{a+}^{\alpha}f\right)(b) + \left(J_{b-}^{\alpha}f\right)(a)\right] \end{aligned}$$

with $\alpha > 0$ where

$$\begin{aligned} WH(t) = \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) \\ \times\left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right)dx. \end{aligned}$$

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then WP is convex and monotonically increasing on $[0,1]$ and*

$$\begin{aligned}&\frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[\left(J_{a^+}^{\alpha}f\right)(b)+\left(J_{b^-}^{\alpha}f\right)(a)\right] \quad (4)\\ =\ & WP(0)\leq WP(t)\leq WP(1)=\frac{f(a)+f(b)}{2}\end{aligned}$$

with $\alpha>0$ *where*

$$\begin{aligned}&WP(t)\\ =\ &\frac{\alpha}{4(b-a)^{\alpha}}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)\right.\\ \times\ &\left(\left(\frac{2b-a-x}{2}\right)^{\alpha-1}+\left(\frac{x-a}{2}\right)^{\alpha-1}\right)\\ +\ &f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\\ \times\ &\left.\left(\left(\frac{b-x}{2}\right)^{\alpha-1}+\left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)\right]dx.\end{aligned}$$

In this study, we establish some refinements of Hermite-Hadamard type inequalities utilizing fractional integrals which generalize the inequalities (2), (3) and (4).

3. Refinements of Hermite Hadamard Type Inequalities

In this section, we will present refinements of Hermite-Hadamard type inequalities via Riemann-Liouville fractional integral operators .

The following Lemma will be frequently used to prove our results.

Lemma 2. *[9] Let* $f:[a,b]\to\mathbb{R}$ *be convex function with* $a<b$ *and* $f\in L[a,b]$. *Let* $A,B,C,D\in[a,b]$ *with* $A+B=C+D$ *and* $|C-D|\leq|A-B|$. *Then,*

$$f(C)+f(D)\leq f(A)+f(B).$$

Theorem 4. *Let* $f:[a,b]\to\mathbb{R}$ *be convex function with* $a<b$ *and* $f\in L[a,b]$. *Let the weight function* $w:[a,b]\to\mathbb{R}$ *be continuous and symmetric about the point* $\left(\frac{a+b}{2},w\left(\frac{a+b}{2}\right)\right)$, *i.e.* $\frac{1}{2}[w(s)+w(a+b-s)]=w\left(\frac{a+b}{2}\right)$. *Then, we have the following inequality*

$$\begin{aligned}&f\left(w\left(\frac{a+b}{2}\right)\right) \quad (5)\\ \leq\ &\frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(w(b))+J_{b^-}^{\alpha}f(w(a))\right]\end{aligned}$$

and if the function w *is monotonic on* $[a,b]$, *then we have*

$$\begin{aligned}&\frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(w(b))+J_{b^-}^{\alpha}f(w(a))\right]\\ \leq\ &\frac{f(w(a))+f(w(b))}{2} \quad (6)\end{aligned}$$

with $\alpha>0$.

Proof. By the hypothesis of symmetricity of the function w, we have

$$2w\left(\frac{a+b}{2}\right)=w(s)+w(a+b-s)$$

and we also have

$$\left|w\left(\frac{a+b}{2}\right)-w\left(\frac{a+b}{2}\right)\right|\leq|w(s)-w(a+b-s)|$$

for $s\in[a,b]$. Applying Lemma 2, we obtain

$$\begin{aligned}&2f\left(w\left(\frac{a+b}{2}\right)\right) \quad (7)\\ \leq\ &f(w(s))+f(w(a+b-s)).\end{aligned}$$

Multiplying by $\frac{(s-a)^{\alpha-1}}{\Gamma(\alpha)}$ both sides of (7) and integrating with respect to s on $[a,b]$, we deduce that

$$\begin{aligned}&\frac{2(b-a)^{\alpha}}{\Gamma(1+\alpha)}f\left(w\left(\frac{a+b}{2}\right)\right)\\ \leq\ &J_{a^+}^{\alpha}f(w(b))+J_{b^-}^{\alpha}f(w(a))\end{aligned}$$

which completes the proof of the inequality (5). By the monotonicity w, we have

$$|w(s)-w(a+b-s)|\leq|w(a)-w(b)|$$

for $s\in[a,b]$ and by symmetricity of the function w, we have

$$w(s)+w(a+b-s)=w(a)+w(b)$$

for $s\in[a,b]$. Applying Lemma 2, we get

$$\begin{aligned}&f(w(s))+f(w(a+b-s)) \quad (8)\\ \leq\ &f(w(a))+f(w(b)).\end{aligned}$$

Multiplying both sides of (8) by $\frac{(s-a)^{\alpha-1}}{\Gamma(\alpha)}$ and integrating with respect to s on $[a,b]$ and dividing both sides by $\frac{2(b-a)^{\alpha}}{\Gamma(1+\alpha)}$, we obtain the desired inequality (6). □

Remark 1. *If we choose $w(t)=t$ in Theorem 4, then the inequalities (5) and (6) reduce to left and right hand sides of the inequality (2), respectively.*

Remark 2. *If we choose $\alpha=1$ in Theorem 4, then Theorem 4 reduces to Theorem 1 proved in [9].*

Theorem 5. *Let the weight function $w:[a,b]\to\mathbb{R}$ be continuous and symmetric about the point $\left(\frac{a+b}{2}, w\left(\frac{a+b}{2}\right)\right)$, i.e. $\frac{1}{2}[w(s)+w(a+b-s)] = w\left(\frac{a+b}{2}\right)$. If $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then WH_w is convex and monotonically increasing on $[0,1]$ and we have the following inequalities*

$$\begin{aligned} & f\left(w\left(\frac{a+b}{2}\right)\right) \qquad (9)\\ = & WH_w(0)\le WH_w(t)\le WH_w(1)\\ = & \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(w(b))+J_{b-}^{\alpha}f(w(a))\right]\end{aligned}$$

with $\alpha>0$ where

$$\begin{aligned} & WH_w(t)\\ = & \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left(tw(x)+(1-t)w\left(\frac{a+b}{2}\right)\right)\\ & \times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx.\end{aligned}$$

Proof. Firstly, for t_1, t_2, $\beta\in[0,1]$, we have

$$\begin{aligned} & WH_w((1-\beta)t_1+\beta t_2)\\ = & \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left(\left(w(x)-w\left(\frac{a+b}{2}\right)\right)\right.\\ & \times[(1-\beta)t_1+\beta t_2]+w\left.\left(\frac{a+b}{2}\right)\right)\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx\\ = & \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left((1-\beta)\right.\\ & \times\left[\left(w(x)-w\left(\frac{a+b}{2}\right)\right)t_1+w\left(\frac{a+b}{2}\right)\right]\\ & \left.+\beta\left(w(x)-w\left(\frac{a+b}{2}\right)\right)t_2+w\left(\frac{a+b}{2}\right)\right)\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx.\end{aligned}$$

Since f is convex, we have

$$\begin{aligned} & WH_w((1-\beta)t_1+\beta t_2)\\ \le & \frac{\alpha(1-\beta)}{2(b-a)^{\alpha}}\\ \times & \int_a^b f\left(\left(w(x)-w\left(\frac{a+b}{2}\right)\right)t_1+w\left(\frac{a+b}{2}\right)\right)\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx\\ + & \frac{\alpha\beta}{2(b-a)^{\alpha}}\\ \times & \int_a^b f\left(\left(w(x)-w\left(\frac{a+b}{2}\right)\right)t_2+w\left(\frac{a+b}{2}\right)\right)\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx\\ = & (1-\beta)WH_w(t_1)+\beta WH_w(t_2).\end{aligned}$$

Hence, we get WH_w is convex on $[0,1]$. On the other hand, we have

$$\begin{aligned} & WH_w(t)\\ = & \frac{\alpha}{2(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} f\left(tw(x)+(1-t)w\left(\frac{a+b}{2}\right)\right)\\ & \times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx\\ & +\frac{\alpha}{2(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^b f\left(tw(x)+(1-t)w\left(\frac{a+b}{2}\right)\right)\\ & \times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx\end{aligned}$$

$$= \frac{\alpha}{2(b-a)^{\alpha}} \times \int_a^{\frac{a+b}{2}} f\left(tw(x)+(1-t)w\left(\frac{a+b}{2}\right)\right) \times \left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx + \frac{\alpha}{2(b-a)^{\alpha}} \times \int_a^{\frac{a+b}{2}} f\left(tw(a+b-x)+(1-t)w\left(\frac{a+b}{2}\right)\right) \times \left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx. \quad (10)$$

Let $t_1 < t_2, t_1, t_2, \in [0,1]$. By the symmetricity of the function w, we have

$$\begin{aligned} &\left[t_1 w(x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right] + \left[t_1 w(a+b-x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right] \\ = &\left[t_2 w(x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right] + \left[t_2 w(a+b-x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right] \end{aligned}$$

and

$$\begin{aligned} &\left|\left[t_1 w(x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right] - \left[t_1 w(a+b-x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right]\right| \\ = &\; t_1\left|w(x)-w(a+b-x)\right| \\ \le &\; t_2\left|w(x)-w(a+b-x)\right| \\ = &\left|\left[t_2 w(x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right] - \left[t_2 w(a+b-x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right]\right| \end{aligned}$$

for $x \in [a,b]$. Hence, applying Lemma 2, we have

$$\begin{aligned} &f\left(t_1 w(x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right) + f\left(t_1 w(a+b-x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right) \\ \le &\; f\left(t_2 w(x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right) + f\left(t_2 w(a+b-x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right). \end{aligned} \quad (11)$$

Multiplying both sides of (11) by

$$\frac{\alpha}{2(b-a)^{\alpha}}\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]$$

and integrating with respect to s on $\left[a, \frac{a+b}{2}\right]$, then by considering the equality (10), we deduce that $WH_w(t_1) \le WH_w(t_2)$. Thus, WH_w is monotonically increasing on $[0,1]$. Using the facts that

$$WH_w(0) = f\left(w\left(\frac{a+b}{2}\right)\right)$$

and

$$WH_w(1) = \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(w(b)) + J_{b^-}^{\alpha} f(w(a))\right]$$

then we obtain the desired result. Thus, the proof is completed. □

Remark 3. *If we choose $w(t) = t$ in Theorem 5, then the inequality (9) reduces to the inequality (3).*

Remark 4. *If we choose $\alpha = 1$ in Theorem 5, then Theorem 5 reduces to Theorem 2 proved in [9].*

Theorem 6. *Let the weight function $w : [a,b] \to \mathbb{R}$ be continuous and monotonic on $[a,b]$ and let w be symmetric about the point $\left(\frac{a+b}{2}, w\left(\frac{a+b}{2}\right)\right)$, i.e. $\frac{1}{2}[w(s)+w(a+b-s)] = w\left(\frac{a+b}{2}\right)$. If $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$, then WP_w is convex and monotonically increasing on $[0,1]$ and we have the following inequalities*

$$\begin{aligned} &\frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(w(b)) + J_{b^-}^{\alpha} f(w(a))\right] \\ = &\; WP_w(0) \le WP_w(t) \le WP_w(1) \\ = &\; \frac{f(w(a))+f(w(b))}{2} \end{aligned} \quad (12)$$

with $\alpha > 0$ where

$$\begin{aligned} & WP_w(t) \\ = & \frac{\alpha}{4(b-a)^\alpha}\int_a^b f\left((1-t)w\left(\frac{a+x}{2}\right)+tw(a)\right) \\ & \times\left(\left(\frac{2b-a-x}{2}\right)^{\alpha-1}+\left(\frac{x-a}{2}\right)^{\alpha-1}\right)dx \\ & +\frac{\alpha}{4(b-a)^\alpha}\int_a^b f\left((1-t)w\left(\frac{x+b}{2}\right)+tw(b)\right) \\ & \times\left(\left(\frac{b-x}{2}\right)^{\alpha-1}+\left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)dx. \end{aligned}$$

Proof. By the way similar to in Theorem, it can be easily proved by convexity of f that WP_w is convex on $[0,1]$. Using change of variable, we have

$$\begin{aligned} & WP_w(t) \qquad (13) \\ = & \frac{\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}} f((1-t)w(s)+tw(a)) \\ & \times\left((b-s)^{\alpha-1}+(u-s)^{\alpha-1}\right)ds \\ & +\frac{\alpha}{2(b-a)^\alpha} \\ & \times\int_a^{\frac{a+b}{2}} f((1-t)w(a+b-s)+tw(b)) \\ & \times\left((b-s)^{\alpha-1}+(s-a)^{\alpha-1}\right)ds. \end{aligned}$$

Let $t_1 < t_2, t_1, t_2, \in [0,1]$. Since w is symmetric to $\frac{a+b}{2}$,

$$w(s)+w(a+b-s)=2w\left(\frac{a+b}{2}\right) \qquad (14)$$

and w is monotonic, we have

$$|w(s)-w(a+b-s)| \le |w(a)-w(b)| \qquad (15)$$

for $s \in [a,b]$. By the equality (14) and the inequality (15), we have

$$\begin{aligned} & [(1-t_1)w(s)+t_1w(a)] \\ & +[(1-t_1)w(a+b-s)+t_1w(b)] \\ = & [(1-t_2)w(s)+t_2w(a)] \\ & +[(1-t_2)w(a+b-s)+t_2w(b)] \end{aligned}$$

and

$$\begin{aligned} & |[(1-t_1)w(s)+t_1w(a)] \\ & -[(1-t_1)w(a+b-s)+t_1w(b)]| \\ = & |(1-t_1)[w(s)-w(a+b-s)] \\ & +t_1[w(a)-w(b)]| \\ \le & (1-t_1)|w(s)-w(a+b-s)| \\ & +t_1|w(a)-w(b)| \\ \le & (1-t_2)|w(s)-w(a+b-s)| \\ & +t_2|w(a)-w(b)| \\ = & |[(1-t_2)w(s)+t_2w(a)] \\ & -[(1-t_2)w(a+b-s)+t_2w(b)]| \end{aligned}$$

for $s \in \left[a, \frac{a+b}{2}\right]$. Therefore, applying Lemma 2, we have

$$\begin{aligned} & f((1-t_1)w(s)+t_1w(a)) \qquad (16) \\ & +f((1-t_1)w(a+b-s)+t_1w(b)) \\ \le & f((1-t_2)w(s)+t_2w(a)) \\ & +f((1-t_2)w(a+b-s)+t_2w(b)). \end{aligned}$$

Multiplying both sides of (16) by

$$\frac{\alpha}{2(b-a)^\alpha}\left[(b-s)^{\alpha-1}+(s-a)^{\alpha-1}\right]$$

and integrating with respect to s on $\left[a, \frac{a+b}{2}\right]$, then by considering the equality (13), we deduce that $WP_w(t_1) \le WP_w(t_2)$. Hence, WP_w is monotonically increasing on $[0,1]$. This completes the proof. □

Remark 5. *If we choose $w(t)=t$ in Theorem 6, then the inequality (12) reduces to the inequality (4).*

Remark 6. *If we choose $\alpha=1$ in Theorem 6, then Theorem 6 reduces to Theorem 3 proved in [9].*

4. Conclusion

In this paper, we present some new weighted refinements of Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals. For further studies we propose to consider the Hermite-Hadamard type inequalities for other fractional integral operators

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RESEARCH ARTICLE

Hermite-Hadamard's inequalities for conformable fractional integrals

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ABSTRACT

In this paper, we establish the Hermite-Hadamard type inequalities for conformable fractional integral and we will investigate some integral inequalities connected with the left and right-hand side of the Hermite-Hadamard type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works and we show that some of our results are better than the other results with respect to midpoint inequalities.

1. Introduction

The convexity property of a given function plays an important role in obtaining integral inequalities. Proving inequalities for convex functions has a long and rich history in mathamatics. In [1], Beckenbach, a leading expert on the theory of convex functions, wrote that the inequality (1) was proved by Hadamard in 1893 [2]. In 1974, Mitrinovič found Hermite and Hadamard's note in Mathesis .

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function define on an interval I of real numbers, and $a, b \in I$ with $a < b$. Then, the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Inequality (1) is known in the literature as Hermite-Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

Over the last decade, classical inequalities have been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, [3–8]

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Definition 1. *The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y) \quad (2)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$.

In [7], Dragomir and Agarwal proved the following results connected with the right part of (1).

Lemma 1. *([7]) Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\\&=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,dt.\end{aligned} \quad (3)$$

Theorem 1. *([7]) Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right).\end{aligned} \quad (4)$$

In [6], Kırmacı gave the following results.

Lemma 2. *([6]) Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ (I° is the interior of I) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\\&=(b-a)\left[\int_0^{1/2}tf'(ta+(1-t)b)\,dt\right.\\&\left.+\int_{1/2}^1(t-1)f'(ta+(1-t)b)\,dt\right].\end{aligned} \quad (5)$$

Theorem 2. *([6]) Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ (I° is the interior of I) with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\alpha}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{b-a}{8}\left(|f'(a)|+|f'(b)|\right).\end{aligned} \quad (6)$$

2. Definitions and Properties of Conformable Fractional Derivative and Integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in [9–14].

Definition 2. ***(Conformable fractional derivative)*** *Given a function $f:[0,\infty)\to\mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_\alpha(f)(t)=\lim_{\varepsilon\to0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon} \quad (7)$$

for all $t>0$, $\alpha\in(0,1]$. If f is $\alpha-$differentiable in some $(0,a)$, $\alpha>0$, $\lim_{t\to0^+}f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\to0^+}f^{(\alpha)}(t). \quad (8)$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

Theorem 3. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then*

i. $D_\alpha(af+bg)=aD_\alpha(f)+bD_\alpha(g)$, for all $a,b\in\mathbb{R}$,

ii. $D_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$,

iii. $D_\alpha(fg)=fD_\alpha(g)+gD_\alpha(f)$,

iv. $D_\alpha\left(\frac{f}{g}\right)=\frac{D_\alpha(f)g-D_\alpha(g)f}{g^2}$.

If f is differentiable, then

$$D_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \quad (9)$$

Also:
1. $D_\alpha(1)=0$
2. $D_\alpha(e^{ax})=ax^{1-\alpha}e^{ax},\ a\in\mathbb{R}$
3. $D_\alpha(\sin(ax))=ax^{1-\alpha}\cos(ax),\ a\in\mathbb{R}$
4. $D_\alpha(\cos(ax))=-ax^{1-\alpha}\sin(ax),\ a\in\mathbb{R}$
5. $D_\alpha\left(\frac{1}{\alpha}t^\alpha\right)=1$
6. $D_\alpha\left(\sin(\frac{t^\alpha}{\alpha})\right)=\cos(\frac{t^\alpha}{\alpha})$

7. $D_\alpha\left(\cos(\frac{t^\alpha}{\alpha})\right) = -\sin(\frac{t^\alpha}{\alpha})$

8. $D_\alpha\left(e^{(\frac{t^\alpha}{\alpha})}\right) = e^{(\frac{t^\alpha}{\alpha})}$.

Theorem 4 (Mean value theorem for conformable fractional differentiable functions)**.** *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ and an α-fractional differentiable mapping on (a,b) with $0 \le a < b$. Then, there exists $c \in (a,b)$, such that*

$$D_\alpha(f)(c) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}}.$$

Definition 3 (Conformable fractional integral)**.** *Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\, d_\alpha x := \int_a^b f(x)\, x^{\alpha-1} dx \tag{10}$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_\alpha^1([a,b])$

Remark 1.

$$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1}f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 5. *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$ we have*

$$I_\alpha^a D_\alpha^a f(t) = f(t) - f(a). \tag{11}$$

Theorem 6. ***(Integration by parts)*** *Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\begin{aligned} &\textstyle\int_a^b f(x)\, D_\alpha^a(g)(x)\, d_\alpha x \\ &= fg|_a^b - \textstyle\int_a^b g(x)\, D_\alpha^a(f)(x)\, d_\alpha x. \end{aligned} \tag{12}$$

Theorem 7. *Assume that $f : [a,\infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have*

$$D_\alpha^a f(t)\, I_\alpha^a = f(t).$$

Theorem 8. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ with $0 \le a < b$. Then,*

$$|I_\alpha^a(f)(x)| \le I_\alpha^a |f|(x).$$

For more details and properties concerning the conformable integral operators, we refer, for example, to the works [15–18].

In this paper, we establish the Hermite-Hadamard type inequalities for conformable fractional integral and we will investigate some integral inequalities connected with the left and right hand side of the Hermite-Hadamard type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works.

3. Hermite-Hadamard's Inequalities for Conformable Fractional Integral

We will start the following important result for α-fractional differentiable mapping;

Theorem 9. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable mapping on (a,b) with $0 \le a < b$. Then, the following conditions are equivalent:*

i) f is a convex functions on $[a,b]$

ii) $D_\alpha f(t)$ is an increasing function on $[a,b]$

iii) for any $x_1, x_2 \in [a,b]$

$$f(x_2) \ge f(x_1) + \frac{(x_2^\alpha - x_1^\alpha)}{\alpha} D_\alpha(f)(x_1). \tag{13}$$

Proof. *i) $\to$ ii)* Let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ and we take $h > 0$ which is small enough such that $x_1 - h, x_2 + h \in [a,b]$. Since $x_1 - h < x_1 < x_2 < x_2 + h$, then we know that

$$\begin{aligned} &\frac{f(x_1) - f(x_1 - h)}{h} \\ &\le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &\le \frac{f(x_2 + h) - f(x_2)}{h}. \end{aligned} \tag{14}$$

Multipling the inequality (14) with $x_1^{1-\alpha} \le x_2^{1-\alpha}$, for $x_1 < x_2$, $\alpha \in (0,1]$, we get

$$\begin{aligned} &x_1^{1-\alpha}\frac{f(x_1) - f(x_1 - h)}{h} \\ &\le x_2^{1-\alpha}\frac{f(x_2 + h) - f(x_2)}{h}. \end{aligned} \tag{15}$$

Let us put $h = \varepsilon x_1^{\alpha-1}$ (and $h = \varepsilon x_2^{\alpha-1}$) such that $h \to 0$, $\varepsilon \to 0$, then the inequality (14) can be converted to

$$\frac{f(x_1) - f(x_1 - \varepsilon x_1^{\alpha-1})}{\varepsilon} \le \frac{f(x_2 + \varepsilon x_2^{\alpha-1}) - f(x_2)}{\varepsilon}.$$

Since f is α-fractional differentiable mapping on (a,b), then let $\varepsilon \to 0^+$, we obtain

$$D_\alpha f(x_1) \le D_\alpha f(x_2) \tag{16}$$

this show that $D_\alpha f$ is increasing in $[a,b]$.

$ii) \to iii)$ Take $x_1, x_2 \in [a,b]$ with $x_1 < x_2$. Since $D_\alpha f$ is increasing in $[a,b]$, then by mean value theorem for conformable fractional differentiable we get

$$\begin{aligned} f(x_2) - f(x_1) &= \frac{(x_2^\alpha - x_1^\alpha)}{\alpha} D_\alpha(f)(c) \\ &\ge \frac{(x_2^\alpha - x_1^\alpha)}{\alpha} D_\alpha(f)(x_1) \end{aligned} \tag{17}$$

where $c \in (x_1, x_2)$. It is follow that

$$f(x_2) \ge f(x_1) + \frac{(x_2^\alpha - x_1^\alpha)}{\alpha} D_\alpha(f)(x_1).$$

$iii) \to i)$ For any $x_1, x_2 \in [a,b]$, we take $x_3 = \lambda x_1 + (1-\lambda)x_2$ and $x_3^\alpha = \lambda x_1^\alpha + (1-\lambda)x_2^\alpha$ for $\lambda \in (0,1)$. It is easy to show that $x_1^\alpha - x_3^\alpha = (1-\lambda)(x_1^\alpha - x_2^\alpha)$ and $x_2^\alpha - x_3^\alpha = -\lambda(x_1^\alpha - x_2^\alpha)$. Thus, by using (13), we obtain that

$$\begin{aligned} f(x_1) &\ge f(x_3) + \frac{(x_1^\alpha - x_3^\alpha)}{\alpha} D_\alpha(f)(x_3) \\ &= f(x_3) + (1-\lambda)\frac{(x_1^\alpha - x_2^\alpha)}{\alpha} D_\alpha(f)(x_3) \end{aligned}$$

and

$$\begin{aligned} f(x_2) &\ge f(x_3) + \frac{(x_2^\alpha - x_3^\alpha)}{\alpha} D_\alpha(f)(x_3) \\ &= f(x_3) - \lambda\frac{(x_1^\alpha - x_2^\alpha)}{\alpha} D_\alpha(f)(x_3). \end{aligned}$$

Both sides of the above two expressions, multiply by λ and $(1-\lambda)$, repectively, and add side to side, then we have

$$\begin{aligned} &\lambda f(x_1) + (1-\lambda) f(x_2) \\ \ge\; & f(x_3) \\ =\; & f(\lambda x_1 + (1-\lambda)x_2) \end{aligned}$$

which is show that f is a convex function. The proof is completed. □

Theorem 10. *Let $\alpha \in (0,1]$, $a \ge 0$, and $f : [a,b] \to \mathbb{R}$ is a continuous function and $\varphi : [0,\infty) \to \mathbb{R}$ be continuous and convex function. Then,*

$$\begin{aligned} &\varphi\left(\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x)\, d_\alpha x\right) \\ &\le \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \varphi(f(x))\, d_\alpha x. \end{aligned} \tag{18}$$

Proof. Let $\varphi : [0,\infty) \to \mathbb{R}$ be a convex function and $x_0 \in [0,\infty)$. From the definition of convexity, there exists $m \in \mathbb{R}$ such that,

$$\varphi(y) - \varphi(x_0) \ge m(y - x_0). \tag{19}$$

Since f is a continuous function

$$x_0 = \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x)\, d_\alpha x \tag{20}$$

is well defined. The function $\varphi \circ f$ is also continuous , thus we may apply (19) with $y = f(t)$ and (20) to obtain

$$\varphi(f(t)) - \varphi(x_0) \ge m(f(t) - x_0).$$

Integrating above inequality from a to b, we get

$$\begin{aligned} &\int_a^b \varphi(f(t)) d_\alpha t - \varphi(x_0)\int_a^b d_\alpha t \\ \ge\; & m\left(\int_a^b f(t) d_\alpha t - x_0 \int_a^b d_\alpha t\right) \\ =\; & m\left(\int_a^b f(t) d_\alpha t - x_0^\alpha \int_a^b d_\alpha t\right) = 0. \end{aligned}$$

It is obvious that the inequality (18) holds. □

Hermite-Hadamard's inequalities can be represented in conformable fractional integral forms as follows:

Theorem 11. *Let $\alpha \in (0,1]$ and $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a convex function and $f \in L_\alpha^1([a^\alpha, b^\alpha])$ with $0 \le a < b$. Then, the following inequality for conformable fractional integral holds:*

$$\begin{aligned} &f\left(\frac{a^\alpha + b^\alpha}{2}\right) \\ &\le \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x \\ &\le \frac{f(a^\alpha) + f(b^\alpha)}{2}. \end{aligned} \tag{21}$$

Proof. Since f is a convex function on $I \subset \mathbb{R}^+$, for $x^\alpha, y^\alpha \in [a^\alpha, b^\alpha]$ with $\lambda = \frac{1}{2}$, we have

$$f\left(\frac{x^\alpha+y^\alpha}{2}\right)\leq\frac{f(x^\alpha)+f(y^\alpha)}{2} \tag{22}$$

i.e, with $x^\alpha = t^\alpha a^\alpha + (1-t^\alpha)b^\alpha$, $y^\alpha = (1-t^\alpha)a^\alpha + t^\alpha b^\alpha$, for $t\in[0,1]$, $\alpha\in(0,1]$

$$\begin{aligned}&2f\left(\frac{a^\alpha+b^\alpha}{2}\right)\\&\leq f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\\&+f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right).\end{aligned} \tag{23}$$

By integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&2\int_0^1 f\left(\tfrac{a^\alpha+b^\alpha}{2}\right)d_\alpha t\\&\leq\int_0^1 f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)d_\alpha t\\&+\int_0^1 f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right)d_\alpha t\\&=\tfrac{2\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x,\end{aligned} \tag{24}$$

and the first inequality is proved. For the proof of the second inequality in (22) we first note that if f is a convex function, then, for $\lambda\in[0,1]$, it yields

$$f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\leq t^\alpha f(a^\alpha)+(1-t^\alpha)f(b^\alpha)$$

and

$$f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right)\leq(1-t^\alpha)f(a^\alpha)+t^\alpha f(b^\alpha).$$

By adding these inequalities we have

$$\begin{aligned}&f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)+f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right)\\&\leq f(a^\alpha)+f(b^\alpha).\end{aligned} \tag{25}$$

Integrating inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)d_\alpha t\\&+\int_0^1 f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right)d_\alpha t\\&\leq\left[f(a^\alpha)+f(b^\alpha)\right]\int_0^1 d_\alpha t\end{aligned}$$

i.e.

$$\frac{1}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x\leq\frac{f(a)+f(b)}{2\alpha}.$$

The proof is completed. □

Remark 2. *If we choose $\alpha=1$ in (21), then inequality (21) become inequality (1).*

Theorem 12. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be a convex function and $f\in L^1_\alpha([a^\alpha,b^\alpha])$ with $0\leq a<b$. Then, for $t\in[0,1]$, the following inequality for conformable fractional integral holds:*

$$\begin{aligned}&f\left(\frac{a^\alpha+b^\alpha}{2}\right)\leq h(t^\alpha)\\&\leq\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x\\&\leq H(t^\alpha)\leq\frac{f(a^\alpha)+f(b^\alpha)}{2}\end{aligned} \tag{26}$$

where

$$\begin{aligned}&h(t^\alpha)=(1-t^\alpha)f\left(\frac{(1+t^\alpha)a^\alpha+(1-t^\alpha)b^\alpha}{2}\right)\\&+t^\alpha f\left(\frac{a^\alpha t^\alpha+(2-t^\alpha)b^\alpha}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&H(t^\alpha)=\tfrac{1}{2}\left[(1-t^\alpha)f(a^\alpha)\right.\\&\left.+f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)+t^\alpha f(b^\alpha)\right].\end{aligned}$$

Proof. Since f is a convex function on I, by applying (21) on the subinterval $[a^\alpha, t^\alpha a^\alpha+(1-t^\alpha)b^\alpha]$, with $t\neq1$, we have

$$\begin{aligned}&f\left(\frac{(1+t^\alpha)a^\alpha+(1-t^\alpha)b^\alpha}{2}\right)\\&\leq\frac{\alpha}{(1-t^\alpha)(b^\alpha-a^\alpha)}\\&\times\int_a^{(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)^{\frac{1}{\alpha}}}f(x^\alpha)\,d_\alpha x\\&\leq\frac{f(a^\alpha)+f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)}{2}.\end{aligned} \tag{27}$$

Now, by applying (21) on the subinterval $[t^\alpha a^\alpha+(1-t^\alpha)b^\alpha, b^\alpha]$, with $t\neq0$, we have

$$\begin{aligned}
&f\left(\frac{a^{\alpha}t^{\alpha}+(2-t^{\alpha})b^{\alpha}}{2}\right)\\
\leq\ & \frac{\alpha}{t^{\alpha}(b^{\alpha}-a^{\alpha})}\int_{(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})^{\frac{1}{\alpha}}}^{b} f(x^{\alpha})\,d_{\alpha}x \qquad (28)\\
\leq\ & \frac{f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})+f(b^{\alpha})}{2}.
\end{aligned}$$

Multiplying (27) by $(1-t^{\alpha})$, and (27) by t^{α}, and adding the resulting inequalities, we obtain the following inequalities

$$h(t^{\alpha})\leq\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}f(x^{\alpha})\,d_{\alpha}x\leq H(t^{\alpha}) \qquad (29)$$

where $h(t^{\alpha})$ and $H(t^{\alpha})$ are defined as in Thereom 12. Using the fact that f is a convex function, we get

$$\begin{aligned}
&f\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)\\
=\ & f\left((1-t^{\alpha})\frac{(1+t^{\alpha})a^{\alpha}+(1-t^{\alpha})b^{\alpha}}{2}\right.\\
&\left.+t^{\alpha}\frac{a^{\alpha}t^{\alpha}+(2-t^{\alpha})b^{\alpha}}{2}\right) \qquad (30)\\
\leq\ & (1-t^{\alpha})f\left(\frac{a^{\alpha}+[t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha}]}{2}\right)\\
&+t^{\alpha}f\left(\frac{[a^{\alpha}t^{\alpha}+(1-t^{\alpha})b^{\alpha}]+b^{\alpha}}{2}\right)\\
\leq\ & \frac{1}{2}\left[(1-t^{\alpha})f(a^{\alpha})\right.\\
&\left.+f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})+t^{\alpha}f(b^{\alpha})\right]\\
\leq\ & \frac{f(a^{\alpha})+f(b^{\alpha})}{2}.
\end{aligned}$$

Therefore, by (29) and (30) we have (26). □

4. Trapezoid Type Inequalities for Conformable Fractional Integral

We need the following lemma. With the help of this, we give some integral inequalities connected with the right-side of Hermite–Hadamard-type inequalities for conformable fractional integral.

Lemma 3. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^{+}\to\mathbb{R}$ be an α-fractional differentiable function on (a,b) with $0\leq a<b$. If $D_{\alpha}(f)$ be an α-fractional integrable function on $[a^{\alpha},b^{\alpha}]$, then the following identity for conformable fractional integral holds:*

$$\begin{aligned}
&\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}f(x^{\alpha})\,d_{\alpha}x-\frac{f(a^{\alpha})+f(b^{\alpha})}{2}\\
&=\frac{1}{2}\int_{0}^{1}(1-2t^{\alpha}) \qquad (31)\\
&\times D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})\,d_{\alpha}t.
\end{aligned}$$

Proof. Integrating by parts

$$\begin{aligned}
&\int_{0}^{1}(1-2t^{\alpha})D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})\,d_{\alpha}t\\
&=(1-2t^{\alpha})f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})\Big|_{0}^{1}\\
&+2\alpha\int_{0}^{1}f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})\,d_{\alpha}t\\
&=-[f(a^{\alpha})+f(b^{\alpha})]+\frac{2\alpha}{(b^{\alpha}-a^{\alpha})}\int_{a}^{b}f(x^{\alpha})\,d_{\alpha}x.
\end{aligned}$$

Thus, by multiplying both sides by $\frac{1}{2}$, we have conclusion (31). □

Remark 3. *If we choose $\alpha=1$ in (31), then equality (31) become equality (3).*

Theorem 13. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^{+}\to\mathbb{R}$ be an α-fractional differentiable function on I° and $D_{\alpha}(f)$ be an α-fractional integrable function on I with $0\leq a<b$. If $|f'|$ be a convex function on I, then the following inequality for conformable fractional integral holds:*

$$\begin{aligned}
&\left|\frac{f(a^{\alpha})+f(b^{\alpha})}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}f(x^{\alpha})\,d_{\alpha}x\right|\\
\leq\ & \frac{\alpha(b^{\alpha}-a^{\alpha})}{2}\left(\frac{2^{3\alpha^{2}}+\left(6\times2^{\alpha^{2}}\right)-8}{3\alpha\times2^{3\alpha^{2}}}\right) \qquad (32)\\
&\left[\frac{a^{\alpha(\alpha-1)}|D_{\alpha}(f)(a^{\alpha})|+b^{\alpha(\alpha-1)}|D_{\alpha}(f)(b^{\alpha})|}{2}\right].
\end{aligned}$$

Proof. Using Lemma 3, it follows that

$$\begin{aligned}
&\left|\frac{f(a^{\alpha})+f(b^{\alpha})}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}f(x^{\alpha})\,d_{\alpha}x\right|\\
&\leq\frac{1}{2}\int_{0}^{1}|1-2t^{\alpha}|\,|D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})|\,d_{\alpha}t.
\end{aligned}$$

Since $|f'|$ is a convex function, by using the properties $D_{\alpha}(f\circ g)(t)=f'(g(t))D_{\alpha}g(t)$ and $D_{\alpha}(f)(t)=t^{1-\alpha}f'(t)$, it follows that

$$\begin{aligned} & |D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)| \\ \leq\ & \alpha(b^\alpha - a^\alpha)\left[t^\alpha a^{\alpha(\alpha-1)} |D_\alpha(f)(a^\alpha)| \right. \\ & \left. + (1-t^\alpha) b^{\alpha(\alpha-1)} |D_\alpha(f)(b^\alpha)|\right] \end{aligned} \tag{33}$$

Using (33), we have

$$\left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x\right|$$

$$\leq \frac{\alpha(b^\alpha - a^\alpha)}{2}\int_0^1 |1-2t^\alpha|$$

$$\times\left[t^\alpha a^{\alpha(\alpha-1)}|D_\alpha(f)(a^\alpha)|\right.$$

$$\left.+(1-t^\alpha) b^{\alpha(\alpha-1)}|D_\alpha(f)(b^\alpha)|\right] d_\alpha t$$

$$= \frac{\alpha(b^\alpha - a^\alpha)}{2}$$

$$\times\left\{a^{\alpha(\alpha-1)}|D_\alpha(f)(a^\alpha)|\int_0^1 |1-2t^\alpha| t^\alpha d_\alpha t\right.$$

$$\left.+ b^{\alpha(\alpha-1)}|D_\alpha(f)(b^\alpha)|\int_0^1 |1-2t^\alpha|(1-t^\alpha)\, d_\alpha t\right\}$$

where

$$\begin{aligned} & \int_0^1 |1-2t^\alpha|(1-t^\alpha)\, d_\alpha t \\ =\ & \int_0^1 |1-2t^\alpha| t^\alpha d_\alpha t = \frac{2^{3\alpha^2} + \left(6\times 2^{\alpha^2}\right) - 8}{3\alpha \times 2^{3\alpha^2}} \end{aligned}$$

Thus, the proof is completed. □

Remark 4. *If we choose $\alpha = 1$ in (32), then inequality (32) become inequality (4).*

Theorem 14. *Let $\alpha \in (0,1]$ and $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be an α-fractional differentiable function on I° and $D_\alpha(f)$ be an α-fractional integrable function on I with $0 \leq a < b$. If $|f'|^q$, $q > 1$, be a convex function on I, then the following inequality for conformable fractional integral holds:*

$$\left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x\right|$$

$$\leq \frac{\alpha(b^\alpha - a^\alpha)}{2}(A(\alpha))^{\frac{1}{p}}$$

$$\left(\frac{a^{q\alpha(\alpha-1)}|D_\alpha(f)(a)|^q + b^{q\alpha(\alpha-1)}|D_\alpha(f)(b)|^q}{2\alpha}\right)^{\frac{1}{q}} \tag{34}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $A(\alpha)$ is given by

$$\begin{aligned} A(\alpha) &= \frac{1}{2\alpha(p+1)}\left\{2 - \left(1 - \frac{1}{2^{\alpha^2-1}}\right)^{p+1}\right. \\ &\quad \left. - \left(\frac{1}{2^{\alpha^2-1}} - 1\right)^{p+1}\right\}. \end{aligned}$$

Proof. Using Lemma 3 and Hölder's integral inequality, we find

$$\left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x\right|$$

$$\leq \frac{1}{2}\int_0^1 |1-2t^\alpha||D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)|\, d_\alpha t$$

$$\leq \frac{1}{2}\left(\int_0^1 |1-2t^\alpha|^p\, d_\alpha t\right)^{\frac{1}{p}}$$

$$\left(\int_0^1 |D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)|^q\, d_\alpha t\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is a convex function, by using the properties $D_\alpha(f \circ g)(t) = f'(g(t))\, D_\alpha g(t)$ and $D_\alpha(f)(t) = t^{1-\alpha} f'(t)$, it follows that

$$\begin{aligned} & |D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)|^q \\ \leq\ & \alpha^q (b^\alpha - a^\alpha)^q \\ & \left[t^\alpha a^{q\alpha(\alpha-1)}|D_\alpha(f)(a^\alpha)|^q\right. \\ & \left. + (1-t^\alpha) b^{q\alpha(\alpha-1)}|D_\alpha(f)(b^\alpha)|^q\right]. \end{aligned} \tag{35}$$

By using (35), we have

$$\left|\frac{f(a^{\alpha})+f(b^{\alpha})}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(x^{\alpha})\,d_{\alpha}x\right|$$
$$\leq \frac{\alpha(b^{\alpha}-a^{\alpha})}{2}\left(\int_0^1 |1-2t^{\alpha}|^p\,d_{\alpha}t\right)^{\frac{1}{p}}$$
$$\left[\int_0^1 \left(t^{\alpha}a^{q\alpha(\alpha-1)}|D_{\alpha}(f)(a^{\alpha})|^q\right.\right.$$
$$\left.\left.+(1-t^{\alpha})\,b^{q\alpha(\alpha-1)}|D_{\alpha}(f)(b^{\alpha})|^q\right)d_{\alpha}t\right]^{\frac{1}{q}}$$
$$\leq \frac{\alpha(b^{\alpha}-a^{\alpha})}{2}\left(\int_0^1 |1-2t^{\alpha}|^p\,d_{\alpha}t\right)^{\frac{1}{p}}$$
$$\left(\frac{a^{q\alpha(\alpha-1)}|D_{\alpha}(f)(a)|^q+b^{q\alpha(\alpha-1)}|D_{\alpha}(f)(b)|^q}{2\alpha}\right)^{\frac{1}{q}}.$$

It follows that

$$\begin{aligned}&\int_0^1 |1-2t^{\alpha}|^p\,d_{\alpha}t\\ =&\int_0^{\frac{1}{2^{\alpha}}}(1-2t^{\alpha})^p\,d_{\alpha}t+\int_{\frac{1}{2^{\alpha}}}^1(2t^{\alpha}-1)^p\,d_{\alpha}t\\ =&\frac{1}{2\alpha(p+1)}\left\{2-\left(1-\frac{1}{2^{\alpha^2-1}}\right)^{p+1}\right.\\ &\left.-\left(\frac{1}{2^{\alpha^2-1}}-1\right)^{p+1}\right\}\end{aligned}$$

which is completed the proof. □

Remark 5. *If we choose $\alpha=1$ in (34), then inequality (34) become Theorem 2.3. in [7].*

5. Midpoint Type Inequalities for Conformable Fractional Integral

We need the following lemma. With the help of this, we give some integral inequalities connected with the left-side of Hermite–Hadamard-type inequalities for conformable fractional integral.

Lemma 4. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be an α-fractional differentiable function on I° with $0\leq a<b$. If $D_{\alpha}(f)$ be an α-fractional integrable function on I, then the following identity for conformable fractional integral holds:*

$$f\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(x^{\alpha})\,d_{\alpha}x$$
$$=\int_0^1 P(t)D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})\,b^{\alpha})\,d_{\alpha}t \tag{36}$$

where

$$P(t)=\begin{cases}t^{\alpha}, & 0\leq t<\frac{1}{2^{1/\alpha}}\\ t^{\alpha}-1, & \frac{1}{2^{1/\alpha}}\leq t\leq 1.\end{cases}$$

Proof. Integrating by parts

$$\begin{aligned}&\int_0^1 P(t)D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})\,b^{\alpha})\,d_{\alpha}t\\ =&\int_0^{\frac{1}{2^{1/\alpha}}} t^{\alpha}D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})\,b^{\alpha})\,d_{\alpha}t\\ &+\int_{\frac{1}{2^{1/\alpha}}}^1 (t^{\alpha}-1)D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})\,b^{\alpha})\,d_{\alpha}t\\ =&\ t^{\alpha}f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})\,b^{\alpha})\Big|_0^{\frac{1}{2^{1/\alpha}}}\\ &-\alpha\int_0^{\frac{1}{2^{1/\alpha}}} f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})\,b^{\alpha})\,d_{\alpha}t\\ &+(t^{\alpha}-1)f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})\,b^{\alpha})\Big|_{\frac{1}{2^{1/\alpha}}}^1\\ &-\alpha\int_{\frac{1}{2^{1/\alpha}}}^1 f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})\,b^{\alpha})\,d_{\alpha}t\\ =&\ f\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)-\frac{\alpha}{(b^{\alpha}-a^{\alpha})}\int_a^b f(x^{\alpha})\,d_{\alpha}x.\end{aligned}$$

Thus, we have conclusion (36). □

Remark 6. *If we choose $\alpha=1$ in (36), then equality (36) become equality (5).*

Theorem 15. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be an α-fractional differentiable function on I° and $D_{\alpha}(f)$ be an α-fractional integrable function on I. If $|f'|$ be a convex function on I, then the following inequality for conformable fractional integrals holds:*

$$\left|\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(x^{\alpha})\,d_{\alpha}x-f\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)\right|$$
$$\leq \frac{\alpha(b^{\alpha}-a^{\alpha})}{8} \tag{37}$$
$$\left(\frac{a^{\alpha(\alpha-1)}|D_{\alpha}(f)(a^{\alpha})|+b^{\alpha(\alpha-1)}|D_{\alpha}(f)(b^{\alpha})|}{\alpha}\right).$$

Proof. Using Lemma 3, it follows that

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x - f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right|$$

$$\leq \left\{\int_0^{\frac{1}{2^{1/\alpha}}} t^\alpha \left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right| d_\alpha t \right.$$

$$\left. + \int_{\frac{1}{2^{1/\alpha}}}^1 (1-t^\alpha)\left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right| d_\alpha t\right\}.$$

By using (33), we have

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x - f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right|$$

$$\leq \alpha(b^\alpha-a^\alpha)\left\{\int_0^{\frac{1}{2^{1/\alpha}}} t^\alpha \left[t^\alpha a^{\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|\right.\right.$$

$$\left. + (1-t^\alpha)b^{\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|\right] d_\alpha t$$

$$+ \int_{\frac{1}{2^{1/\alpha}}}^1 (1-t^\alpha)\left[t^\alpha a^{\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|\right.$$

$$\left.\left. + (1-t^\alpha)b^{\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|\right] d_\alpha t\right\}$$

$$= \frac{\alpha(b^\alpha-a^\alpha)}{8} \times \left(\frac{a^{\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right| + b^{\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|}{\alpha}\right).$$

Thus, the proof is completed. □

Remark 7. *If we choose $\alpha = 1$ in (37), then inequality (37) become the inequality (6).*

Theorem 16. *Let $\alpha \in (0,1]$ and $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be an α-fractional differentiable function on I° and $D_\alpha(f)$ be an α-fractional integrable function on I. If $|f'|^q$, $q > 1$,be a convex function on I, then the following inequality for conformable fractional integrals holds:*

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x - f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right| \tag{38}$$

$$\leq \alpha(b^\alpha-a^\alpha)\left(\frac{1}{\alpha(p+1)2^{p+1}}\right)^{1/p} B(\alpha)$$

where $\frac{1}{p}+\frac{1}{q}=1$, $B(\alpha)$ is defined by

$$B(\alpha) = \left(\frac{a^{q\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|^q}{8\alpha} + \frac{3b^{q\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|}{8\alpha}\right)^{1/q}$$

$$+ \left(\frac{3a^{q\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|^q}{8\alpha} + \frac{b^{q\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|}{8\alpha}\right)^{1/q}.$$

Proof. Using Lemma 3 and from Hölder's inequality, it follows that

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x - f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right|$$

$$\leq \left\{\int_0^{\frac{1}{2^{1/\alpha}}} t^\alpha \left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right| d_\alpha t \right.$$

$$\left. + \int_{\frac{1}{2^{1/\alpha}}}^1 (1-t^\alpha)\left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right| d_\alpha t\right\}$$

$$\leq \left\{\left(\int_0^{\frac{1}{2^{1/\alpha}}} t^{p\alpha} d_\alpha t\right)^{1/p}\right.$$

$$\times \left(\int_0^{\frac{1}{2^{1/\alpha}}} \left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right|^q d_\alpha t\right)^{1/q}$$

$$+ \left(\int_{\frac{1}{2^{1/\alpha}}}^1 (1-t^\alpha)^p d_\alpha t\right)^{1/p}$$

$$\left.\left(\int_{\frac{1}{2^{1/\alpha}}}^1 \left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right|^q d_\alpha t\right)^{1/q}\right\}.$$

By using (35), it follows that

$$\left|\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b f(x^{\alpha})\,d_{\alpha}x - f\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)\right|$$

$$\leq \alpha(b^{\alpha}-a^{\alpha})\left(\frac{1}{\alpha(p+1)2^{p+1}}\right)^{1/p}$$

$$\times\left\{\left(\int_0^{\frac{1}{2^{1/\alpha}}}\left[t^{\alpha}a^{q\alpha(\alpha-1)}\left|D_{\alpha}(f)(a^{\alpha})\right|^q\right.\right.\right.$$

$$\left.\left.+(1-t^{\alpha})\,b^{q\alpha(\alpha-1)}\left|D_{\alpha}(f)(b^{\alpha})\right|^q\right]d_{\alpha}t\right)^{1/q}$$

$$+\left(\int_{\frac{1}{2^{1/\alpha}}}^{1}\left[t^{\alpha}a^{q\alpha(\alpha-1)}\left|D_{\alpha}(f)(a^{\alpha})\right|^q\right.\right.$$

$$\left.\left.\left.+(1-t^{\alpha})\,b^{q\alpha(\alpha-1)}\left|D_{\alpha}(f)(b^{\alpha})\right|^q\right]d_{\alpha}t\right)^{1/q}\right\}$$

$$=\alpha(b^{\alpha}-a^{\alpha})\left(\frac{1}{\alpha(p+1)2^{p+1}}\right)^{1/p}$$

$$\times\left\{\left(\frac{a^{q\alpha(\alpha-1)}\left|D_{\alpha}(f)(a^{\alpha})\right|^q}{8\alpha}\right.\right.$$

$$\left.+\frac{3b^{q\alpha(\alpha-1)}\left|D_{\alpha}(f)(b^{\alpha})\right|}{8\alpha}\right)^{1/q}$$

$$+\left(\frac{3a^{q\alpha(\alpha-1)}\left|D_{\alpha}(f)(a^{\alpha})\right|^q}{8\alpha}\right.$$

$$\left.\left.+\frac{b^{q\alpha(\alpha-1)}\left|D_{\alpha}(f)(b^{\alpha})\right|}{8\alpha}\right)^{1/q}\right\}.$$

Thus, the proof of completed. □

Remark 8. *If we choose $\alpha = 1$ in (38), then inequality (38) become the inequality (2.1) in Theorem 2.3. in [6].*

6. Conclusion

In this work, we have obtained some new Hermite-Hadamard type integral inequalities for conformable integrals and we will investigate some integral inequalities connected with the left and right hand side of the Hermite-Hadamard type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works and we show that some our results are better than the other results with respect to midpoint inequalities.

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RESEARCH ARTICLE

On stable high order difference schemes for hyperbolic problems with the Neumann boundary conditions

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ABSTRACT

In this paper, third and fourth order of accuracy stable difference schemes for approximately solving multipoint nonlocal boundary value problems for hyperbolic equations with the Neumann boundary conditions are considered. Stability estimates for the solutions of these difference schemes are presented. Finite difference method is used to obtain numerical solutions. Numerical results of errors and CPU times are presented and are analyzed.

1. Introduction

Many mathematical models of natural and applied sciences phenomena such as fluid mechanics, hydrodynamics, electromagnetics and various areas of physics are based on hyperbolic partial differential equations. Modeling some of these phenomena, imposing nonlocal conditions may be more accurate than classical conditions. Nonlocal boundary condition is a relation between the values of unknown function on the boundary and inside of the given domain. Over the last decades, boundary value problems with nonlocal boundary conditions have become a rapidly growing area of research. Such types of boundary conditions are encountered in applications including thermoelasticity [1], climate control systems [2] and financial mathematics [3]. Boundary value problems for parabolic, elliptic and equations of mixed types are actively studied by many scientists for decades (see [4]- [27]). Stability has been an important research area in the development of numerical methods. Particulary, in this work stability analysis is performed by suitable unconditionally stable difference schemes with an unbounded operator. Some results of this paper, without proof, are presented in [27].

In the present paper, third and fourth order of accuracy stable difference schemes for approximately solving the multipoint nonlocal boundary value problem (NBVP)

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} = f(t,x), \\ x = (x_1, \dots, x_m) \in \Omega,\ 0 < t < 1, \\ u(0,x) = \sum\limits_{j=1}^{n} \alpha_j u(\lambda_j, x) + \varphi(x), x \in \overline{\Omega}, \\ u_t(0,x) = \sum\limits_{j=1}^{n} \beta_j u_t(\lambda_j, x) + \psi(x), x \in \overline{\Omega} \end{cases} \quad (1)$$

for the multidimensional hyperbolic equation with the Neumann boundary condition

$$\frac{\partial u(t,x)}{\partial \vec{n}}|_{x \in S} = 0, x \in S$$

or mixed conditions

$$u(t,x)|_{x \in S_1} = 0,\ \frac{\partial u(t,x)}{\partial \vec{n}}|_{x \in S_2} = 0,$$

$$x \in S, S = S_1 \cup S_2$$

are considered.

Here

$$\Omega = \{x = (x_1, \cdots, x_m) : 0 < x_j < 1, 1 \leq j \leq m\}$$

is the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m$, with boundary S, $\overline{\Omega} = \Omega \cup S$ and $a_r(x)$ $(a_r(x) \geq a > 0, x \in \Omega)$, $\varphi(x), \psi(x)$ $\left(x \in \overline{\Omega}\right)$, $f(t,x)$ $(t \in (0,1), x \in \Omega)$ are given smooth functions.

2. Stability Estimates for High Order Difference Schemes

In the present section the third and the fourth order absolutely stable difference schemes and stability estimates for the solutions of these difference schemes are presented. These difference schemes are obtained in [18]. The discretization of problem (1) with Neumann condition or mixed conditions is carried out in two steps. In the first step, the grid sets are defined as

$$\widetilde{\Omega}_h = \{x = x_r = (h_1 r_1, \ldots, h_m r_m),$$

$$r = (r_1, \cdots, r_m), 0 \leq r_j \leq N_j,$$

$$h_j N_j = 1, j = 1, \cdots, m\},$$

$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S,$$

and difference operator A_h^x is given by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{m} \left(a_r(x) u_{\overline{x}_r}^h\right)_{x_r, j_r} \tag{2}$$

acting in the space of grid functions $u^h(x)$ for all $x \in S_h$. Note that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$ with the domain $D\left(A_h^x\right) = \left\{u\left(x\right) \in W_{2h}^2\left(\widetilde{\Omega}_h\right), \frac{\partial u}{\partial \overrightarrow{n}} = 0 \text{ on } S_h\right\}$.

The spaces $L_{2h} = L_2(\widetilde{\Omega}_h)$, $W_{2h}^1 = W_{2h}^1\left(\widetilde{\Omega}_h\right)$ and $W_{2h}^2 = W_{2h}^2\left(\widetilde{\Omega}_h\right)$ of the grid functions

$$\varphi^h(x) = \{\varphi(h_1 r_1, \ldots, h_m r_m)\}$$

are defined on $\widetilde{\Omega}_h$, equipped with norms

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \ldots h_m\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^1} = \left\|\varphi^h\right\|_{L_{2h}}$$

$$+ \left(\sum_{x \in \overline{\Omega_h}} \sum_{r=1}^{m} \left|\left(\varphi^h\right)_{\overline{x}_r, j_r}\right|^2 h_1 \ldots h_m\right)^{1/2},$$

and

$$\left\|\varphi^h\right\|_{W_{2h}^2} = \left\|\varphi^h\right\|_{L_{2h}}$$

$$+ \left(\sum_{x \in \overline{\Omega}_h} \sum_{r=1}^{m} \left|\left(\varphi^h\right)_{\overline{x}_r}\right|^2 h_1 \ldots h_m\right)^{1/2}$$

$$+ \left(\sum_{x \in \overline{\Omega}_h} \sum_{r=1}^{m} \left|\left(\varphi^h\right)_{x_r \overline{x}_r, j_r}\right|^2 h_1 \ldots h_m\right)^{1/2},$$

respectively.

Using difference operator A_h^x the following NBVP

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x), \\ 0 < t < 1, \ x \in \Omega_h, \\ v^h(0,x) = \sum\limits_{j=1}^{n} \alpha_j v^h\left(\lambda_j, x\right) + \varphi^h(x), x \in \widetilde{\Omega}_h, \\ \frac{dv^h(0,x)}{dt} = \sum\limits_{j=1}^{n} \beta_j v_t^h(\lambda_j, x) + \psi^h(x), x \in \widetilde{\Omega}_h \end{cases} \tag{3}$$

is obtained.

In the next step problem (3) is replaced by the third order of accuracy difference scheme

$$\begin{cases} \tau^{-2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + \frac{2}{3} A_h^x u_k^h(x) \\ + \frac{1}{6} A_h^x\left(u_{k+1}^h(x) + u_{k-1}^h(x)\right) + \frac{1}{12}\tau^2\left(A_h^x\right)^2 u_{k+1}^h(x) \\ = f_k^h(x), f_k^h(x) = \frac{2}{3} f^h(t_k, x) + \frac{1}{6}\left(f^h(t_{k+1}, x)\right) \\ + f^h(t_{k-1}, x)\big) - \frac{1}{12}\tau^2\left(-Af^h(t_{k+1}, x) + f_{tt}^h(t_{k+1}, x)\right), \\ t_k = k\tau, 1 \leq k \leq N-1, N\tau = 1, x \in \Omega_h, \\ u_0^h(x) = \sum\limits_{j=1}^{n} \alpha_j \left\{u_{[\lambda_j/\tau]}^h(x)\right. \\ + \tau^{-1}\left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)\left(\lambda_j - [\lambda_j/\tau]\tau\right) \\ + \frac{3}{2}\left(f_{[\lambda_j/\tau]} - A_h^x u_{[\lambda_j/\tau]}^h(x)\right)\left(\lambda_j - [\lambda_j/\tau]\tau\right)^2 \\ + \frac{7}{6}\left(f'_{[\lambda_j/\tau]} - \tau^{-1} A_h^x\left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)\right) \\ \left.\times\left(\lambda_j - [\lambda_j/\tau]\tau\right)^3\right\} + \varphi^h(x), x \in \Omega_h, \\ \left(I + \tau^2\left(A_h^x\right)^4\right)\tau^{-1}\left(u_1^h(x) - u_0^h(x)\right) \\ = \sum\limits_{j=1}^{n} \beta_j \left\{\tau^{-1}\left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)\right. \\ + \left(f_{[\lambda_j/\tau]} - A_h^x u_{[\lambda_j/\tau]}^h(x)\right)\left(\lambda_j - [\lambda_j/\tau]\tau\right) \\ + \frac{1}{2!}\left(f'_{[\lambda_j/\tau]} - \tau^{-1} A_h^x\left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)\right) \\ \times\left(\lambda_j - [\lambda_j/\tau]\tau\right)^2 + \frac{1}{3!}\left(f''_{[\lambda_j/\tau]} - A_h^x f_{[\lambda_j/\tau]}\right. \\ \left. + \left(A_h^x\right)^2 u_{[\lambda_j/\tau]}^h(x)\right)\left(\lambda_j - [\lambda_j/\tau]\tau\right)^3\Big\} + \psi^h(x), \\ x \in \Omega_h, f_{1,1}^h\left(x\right) = \frac{1}{2} f^h\left(0, x\right) + \frac{\tau}{6} f_t^h\left(0, x\right). \end{cases} \tag{4}$$

Theorem 1. *Let τ and $|h|$ be sufficiently small numbers. Then, the solution of difference scheme (4) satisfies the following stability estimates:*

$$\max_{0\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^1}$$
$$\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\varphi^h\right\|_{W_{2h}^1}\right.$$
$$\left.+\tau\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{1,1}^h\right\|_{L_{2h}}\right],$$
$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}$$
$$+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^2}\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}\right.$$
$$+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}^1}$$
$$\left.+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|\varphi^h\right\|_{W_{2h}^3}+\tau\left\|f_{1,1}^h\right\|_{W_{2h}^1}\right]$$

where M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x)$, $f_{1,1}^h$ and $f_k^h, 1\le k<N$.

This theorem is proved in [25] under the following assumption

$$\sum_{k=1}^{n}|\alpha_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.$$
$$\left.+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$
$$+\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.$$
$$\left.+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$
$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.$$
$$\left.+\frac{7}{12}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4+\frac{7}{36}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6\right\}<1. \tag{5}$$

In the third step replacing problem (3) by the fourth order of accuracy difference scheme problem

$$\begin{cases}
\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{5}{6}A_h^x u_k^h(x)\\
+\frac{1}{12}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)-\frac{1}{72}\tau^2\left(A_h^x\right)^2u_k^h(x)+\frac{\tau^2}{144}\left(A_h^x\right)^2\\
\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)=f_k^h(x),\ f_k^h(x)=\frac{5}{6}f^h(t_k,x)\\
+\frac{1}{12}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)+\frac{\tau^2}{72}\left(-A_h^xf^h(t_k,x)+f_{tt}^h(t_k,x)\right)\\
-\frac{\tau^2}{144}\left(-A_h^x\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)\right.\\
\left.+f_{tt}^h(t_{k+1},x)+f_{tt}^h(t_{k-1},x)\right),x\in\Omega_h,\\
t_k=k\tau,1\le k\le N-1,N\tau=1,\\
u_0^h(x)=\left(I-\frac{i\tau}{2}\left(A_h^x\right)^{1/2}+\frac{\tau^2}{12}\left(A_h^x\right)^3\right)^{-1}\\
\sum_{k=1}^{n}\alpha_k\left\{\left(\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)-\frac{7\tau^2}{6}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3\right)\right.\\
\times\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)+\left(1-\frac{3\tau^2}{2}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2\right.\\
\left.+\frac{\tau^4}{24}\left(A_h^x\right)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\right)u_{[\lambda_j/\tau]}^h+\frac{3\tau^2}{2}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2f_{[\lambda_j/\tau]}\\
+\frac{7\tau^3}{6}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3f'_{[\lambda_j/\tau]}+\frac{\tau^4}{24}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4f''_{[\lambda_j/\tau]}\\
\left.-\frac{\tau^4}{24}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4f_{[\lambda_j/\tau]}\right\}+\varphi^h(x),x\in\widetilde{\Omega}_h,\\
\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right)\\
=\left(I-\frac{\tau^2}{12}A_h^x\right)\left(I+\frac{i\tau}{2}\left(A_h^x\right)^{1/2}+\frac{\tau^2}{12}\left(A_h^x\right)^3\right)^{-1}\\
\times\sum_{k=1}^{n}\beta_k\left\{\left(\frac{1}{\tau}-\frac{\tau}{2}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2\right.\right.\\
\left.+\frac{\tau^3}{24}\left(A_h^x\right)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\right)\left(u_{[\lambda_j/\tau]}-u_{[\lambda_j/\tau]-1}\right)\\
+\left(-A_h^x\tau\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)+\frac{\tau^3}{6}\left(A_h^x\right)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3\right)u_{[\lambda_j/\tau]}\\
+\tau f_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)+\frac{\tau^2}{2}f'_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2\\
+\frac{\tau^3}{6}f''_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3+\frac{\tau^4}{24}f'''_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\\
-\frac{\tau^3}{6}A_h^xf_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3\\
\left.-\frac{\tau^4}{24}A_h^xf'_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\right\}+\psi^h(x),x\in\widetilde{\Omega}_h.
\end{cases} \tag{6}$$

is obtained.

Theorem 2. *Let τ and h be sufficiently small numbers. Then, solution of difference scheme (6) obeys the following stability estimates:*

$$\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^1}+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{L_{2h}}$$
$$\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}\right.$$
$$\left.+\left\|\varphi^h\right\|_{W_{2h}^1}+\tau\left\|f_{2,2}^h\right\|_{L_{2h}}\right],$$
$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}$$
$$+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{W_{2h}^1}$$
$$+\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}\right.$$
$$+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}$$
$$\left.+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{2,2}^h\right\|_{W_{2h}^1}\right].$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x)$, $f^h_{2,2}$ and $f^h_k, 1 \le k < N$.

This theorem is proved in [25] under the following assumption

$$\left\{\sum_{k=1}^{n}|\alpha_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2 +\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\} +\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2 +\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\} +\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2 +\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4+\frac{1}{9}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6 +\frac{1}{576}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^8\right\} < 1. \qquad (7)$$

3. Numerical Analysis

In the present section some examples are presented to verify theoretical statements. Finite difference method is used and symbolic computations are carried out by Matlab. Three problems for one dimensional hyperbolic equations with the Neumann boundary conditions and mixed type boundary conditions are considered. Results of numerical experiments are presented in tables and are analyzed.
The grid set $[0,1]_\tau\times\ [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h with

$$[0,1]_\tau\times[0,\pi]_h=\{(t_k,x_n): t_k=k\tau, 0\le k\le N,$$

$$N\tau=1, x_n=nh,\ \ 0\le n\le M, Mh=\pi\}$$

is considered.

Example 1. *Let us consider problem*

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial^2 u(t,x)}{\partial x^2}=e^{-t}(\sin^2 x-2\cos 2x),\\ 0<t<1, 0<x<\pi,\\ u(0,x)=\frac{1}{10}u(1,x)+\frac{1}{10}u(\frac{1}{2},x)\\ +(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x, 0\le x\le\pi,\\ u_t(0,x)=\frac{1}{10}u_t(1,x)+\frac{1}{10}u_t(\frac{1}{2},x)\\ +(-1+\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x, 0\le x\le\pi,\\ u_x(t,0)=u_x(t,\pi)=0 \end{cases} \qquad (8)$$

for one-dimensional hyperbolic equation with constant coefficients.

The exact solution of this problem is

$$u\left(t,x\right)=e^{-t}\sin^2 x.$$

In approximately solving problem (8), third and fourth order of accuracy difference schemes (4) and (6) are used respectively.

In the first step, applying simple formulas

$$\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2}-u^{''}\left(x_n\right)=O\left(h^2\right), \quad (9)$$

$$\frac{35u(0)-104u(0+\tau)+114u(0+2\tau)-56u(0+3\tau)+11u(0+4\tau)}{12\tau^2}$$

$$-u^{''}\left(0\right)=O\left(\tau^3\right), \qquad (10)$$

$$\frac{-5u(0)+18u(h)-24u(2h)+14u(3h)-3u(4h)}{2\tau^3}$$

$$-u^{'''}\left(0\right)=O\left(\tau^4\right), \qquad (11)$$

and using difference scheme (4), the second order of accuracy in t third order of accuracy in x difference scheme

$$
\begin{cases}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{2}{3}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right) \\
-\frac{1}{6}\left(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\
+\frac{\tau^2}{12}\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}\right)=\varphi_n^k, \\
\varphi_n^k=\left\{\frac{2}{3}e^{-t_k}+\frac{1}{6}(e^{-t_{k+1}}+e^{-t_{k-1}})\right. \\
\left.-\frac{\tau^2}{12}e^{-t_{k+1}}\right\}\sin^2 x_n-2\left\{\frac{2}{3}e^{-t_k}\right. \\
\left.+\frac{1}{6}(e^{-t_{k+1}}+e^{-t_{k-1}})+\frac{\tau^2}{3}e^{-t_{k+1}}\right\}\cos 2x_n, \\
t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\
x_n=nh, 2\le n\le M-2, Mh=\pi, \\
u_n^0-\frac{1}{8}u_n^{(N/2)}-\frac{1}{8}u_n^N \\
=(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x_n, 0\le n\le M, \\
(u_n^1-u_n^0) \\
-\frac{\tau^2}{12}\left(\frac{(u_{n+1}^1-u_{n+1}^0)-2(u_n^1-u_n^0)+(u_{n-1}^1-u_{n-1}^0)}{h^2}\right) \\
+\frac{\tau^4}{144}\left[\frac{(u_{n+2}^1-u_{n+2}^0)-4(u_{n+1}^1-u_{n+1}^0)+6(u_n^1-u_n^0)}{h^4}\right. \\
\left.+\frac{-4(u_{n-1}^1-u_{n-1}^0)+(u_{n-2}^1-u_{n-2}^0)}{h^4}\right]=\varphi_n^N, \\
\varphi_n^N=(-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{6}+\frac{\tau^4}{6})\sin^2 x \\
+(\frac{\tau^3}{6}+\frac{\tau^4}{12}+\frac{35}{36}\tau^5-\frac{5}{18}\tau^6-\frac{5}{54}\tau^7)\cos 2x \\
+(\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}}),\quad 2\le n\le M-2
\end{cases}
\tag{12}
$$

for the approximate solution of problem (8) is obtained. By rearranging like terms of the problem, the following linear system

$$AU_{n+2}+BU_{n+1}+CU_n+DU_{n-1}+EU_{n-2}=R\varphi_n, \tag{13}$$

$$2\le n\le M-2$$

with $(N+1)\times(N+1)$ matrix coefficients

$$
A=\begin{bmatrix}
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & x & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & x & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & x & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & x & 0 \\
-r & r & 0 & 0 & \dots & 0 & 0 & 0
\end{bmatrix},
$$

$$
B=\begin{bmatrix}
0 & 0 & 0 & \dots & 0 & 0 & 0 \\
y & w & v & \dots & 0 & 0 & 0 \\
0 & y & w & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & w & v & 0 \\
0 & 0 & 0 & \dots & y & w & v \\
s & -s & 0 & \dots & 0 & 0 & 0
\end{bmatrix},
$$

$$
C=\begin{bmatrix}
1 & 0 & 0 & \dots 0 & \frac{-1}{8} & 0\dots & 0 & \frac{-1}{8} \\
l & n & m & 0 & \dots & 0 & 0 & 0 \\
0 & l & n & m & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \ddots & n & m & 0 \\
0 & 0 & 0 & \dots & \dots & l & n & m \\
-t & t & 0 & \dots & \dots & 0 & 0 & 0
\end{bmatrix},
$$

$$D=B,\ E=A,$$

$$
R=\begin{bmatrix}
1 & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & 0 & \dots & 0 & 1
\end{bmatrix},
$$

where the entries are

$$x=\frac{\tau^2}{12h^4}, v=-\frac{1}{6h^2}-\frac{\tau^2}{3h^4}, w=-\frac{2}{3h^2},$$

$$y=-\frac{1}{6h^2}, m=\frac{1}{\tau^2}+\frac{1}{3h^2}+\frac{\tau^2}{2h^4},$$

$$n=-\frac{2}{\tau^2}+\frac{4}{3h^2}, l=\frac{1}{\tau^2}+\frac{1}{3h^2},$$

$$r=\frac{\tau^4}{144h^4}, s=\frac{\tau^2}{12h^2}+\frac{\tau^4}{36h^4}$$

$$t=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4},$$

and $(N+1)\times 1$ column matrices

$$
\varphi_n^k=\begin{bmatrix}
\varphi_n^0 \\
\varphi_n^1 \\
\vdots \\
\varphi_n^N
\end{bmatrix}_{(N+1)\times 1}, 0\le k\le N,
$$

with

$$\varphi_n^0=(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2(x_n), 0\le n\le M,$$

$$
\begin{aligned}
\varphi_n^N &= \{-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{6}+\frac{\tau^4}{6}\}\sin^2(x_n) \\
&+\left\{\frac{\tau^3}{6}+\frac{\tau^4}{12}+\frac{35}{36}\tau^5\right. \\
&\left.-\frac{5}{18}\tau^6-\frac{5}{54}\tau^7\right\}\cos 2x_n \\
&+(\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2(x_n)
\end{aligned}
$$

$$\varphi_n^k = \left\{\frac{2}{3}e^{-t_k} + \frac{1}{6}(e^{-t_{k+1}} + e^{-t_{k-1}}) - \frac{\tau^2}{12}e^{-t_{k+1}}\right\}\sin^2 x_n,$$

$$+2\left\{\frac{2}{3}e^{-t_k} + \frac{1}{6}(e^{-t_{k+1}} + e^{-t_{k-1}}) + \frac{\tau^2}{3}e^{-t_{k+1}}\right\}\cos 2x_n,$$

$$1 \le k \le N-1,$$

$$U_s^k = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

$$0 \le k \le N, s = n-2, n-1, n, n+1, n+2$$

is obtained.

The modified Gauss elimination method is used and the following formula

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1},$$

$$n = M-2, ...2, 1, 0$$

is applied where α_j, β_j $(j = 1, \ldots, M)$ are $(N+1)\times(N+1)$ square matrices and γ_j are $(N+1)\times 1$ column matrices for the solution of difference scheme (12). From that one can obtain formulas $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$

$$\begin{cases} \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}A, \\ \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1} \\ \times(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \gamma_{n+1} = +(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1} \\ \times(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}), \end{cases} \tag{14}$$

where $n = 2 : M-2$ and

$$\gamma_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\alpha_2 = \begin{bmatrix} 4/5 & 0 & \ldots & 0 \\ 0 & 4/5 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 4/5 \end{bmatrix},$$

$$\beta_2 = \begin{bmatrix} -1/5 & 0 & \ldots & 0 \\ 0 & -1/5 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1/5 \end{bmatrix}.$$

In a similar manner the following formulas

$$U_M = -[P + Q(4I - \alpha_{M-1})^{-1}(\beta_{M-1} + 3I)]^{-1} \times \{R + Q(4I - \alpha_{M-1})^{-1}\gamma_{M-1}\} \tag{15}$$

$$U_{M-1} = -(P+Q)^{-1}R \tag{16}$$

$$U_{M-2} = (4I - \alpha_{M-2})^{-1} \times \{(5I + \beta_{M-2})U_{M-1} + \gamma_{M-2}\}, \tag{17}$$

where

$$P = \frac{1}{6h}(11I + 9\beta_{M-1} - 2\alpha_{M-2}\beta_{M-1}),$$

$$Q = \frac{1}{6h}(-18I + 9\alpha_{M-1} - 2(\alpha_{M-2}\alpha_{M-1} + \beta_{M-2}))$$

$$R = \frac{1}{6h}(9\gamma_{M-1} - 2\alpha_{M-2}\gamma_{M-1} - 2\gamma_{M-2})$$

are obtained. The system

$$U_0 = \alpha_1 U_1 + \beta_1 U_2 + \gamma_1 \tag{18}$$

where

$$\alpha_1 = \frac{-1}{h}T^{-1}, \beta_1 = 0, \gamma_1 = \frac{h}{2}T^{-1}\varphi_n^0$$

is used for the boundary condition $u_x(t, 0) = 0$ of third order of accuracy difference scheme. Here

$$T = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & 0 & \ldots & 0 \\ a & b & a & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & a & b & a & 0 & \ldots & \ldots & \vdots \\ \vdots & 0 & a & b & a & 0 & \ldots & \vdots \\ \vdots & \ldots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ldots & \ldots & 0 & a & b & a & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & a & b & a \\ 0 & \ldots & 0 & \lambda_5 & \lambda_4 & \lambda_3 & \lambda_2 & \lambda_1 \end{bmatrix} \tag{19}$$

with

$$
\begin{aligned}
\lambda_1 &= \left(-\frac{1}{h}-\frac{35h}{24\tau^2}+\frac{5h^2}{12\tau^3}\right),\\
\lambda_2 &= \left(\frac{104h}{24\tau^2}-\frac{18h^2}{12\tau^3}\right),\\
\lambda_3 &= \left(-\frac{114h}{24\tau^2}+\frac{24h^2}{12\tau^3}\right),\\
\lambda_4 &= \left(\frac{56h}{24\tau^2}-\frac{14h^2}{12\tau^3}\right),\\
\lambda_5 &= \left(-\frac{11h}{24\tau^2}+\frac{3h^2}{12\tau^3}\right),\\
a &= -\frac{h}{2\tau^2}\ ,\ \ b=\left(-\frac{1}{h}+\frac{h}{\tau^2}\right).
\end{aligned}
$$

In the next step difference scheme (6) and the formulas

$$\frac{-3u\left(1\right)+4u\left(1-h\right)-u\left(1-2h\right)}{2h}-u^{'}\left(1\right)=O\left(h^2\right),$$

$$\frac{1}{4\tau^3}\left(-17u\left(0\right)+71u\left(0+\tau\right)-118u\left(0+2\tau\right)\right.$$

$$\left.+98u\left(0+3\tau\right)-41u\left(0+4\tau\right)+7u(0+5\tau)\right)$$

$$-u^{'''}\left(0\right)=O\left(\tau^3\right),$$

$$\frac{u\left(0\right)-2u\left(0+\tau\right)+u\left(0+2\tau\right)}{\tau^2}-u^{''}\left(0\right)=O\left(\tau^3\right)$$

are used to obtain second order of accuracy in t and fourth order of accuracy in x difference scheme

$$
\begin{cases}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{5}{6}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right)\\
-\frac{1}{12}\left(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right)\\
-\frac{\tau^2}{72}\left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{h^4}\right)\\
+\frac{\tau^2}{144}\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}\right.\\
\left.+\frac{u_{n+2}^{k-1}-4u_{n+1}^{k-1}+6u_n^{k-1}-4u_{n-1}^{k-1}+u_{n-2}^{k-1}}{h^4}\right)=\varphi_n^k,\\
\varphi_n^k=\left\{(\frac{5}{6}+\frac{\tau^2}{72})e^{-t_k}\right.\\
\left.+(\frac{1}{12}-\frac{\tau^2}{144})(e^{-t_{k+1}}+e^{-t_{k-1}})\right\}\sin^2(x_n)\\
+\left\{(-\frac{5}{3}+\frac{\tau^2}{9})e^{-t_k}\right.\\
\left.-(\frac{1}{6}+\frac{\tau^2}{18})(e^{-t_{k+1}}+e^{-t_{k-1}})\right\}\cos 2x_n\\
t_k=k\tau,\ 1\le k\le N-1, N\tau=1,\\
x_n=nh, 1\le n\le M-1, Mh=\pi,\\
\varphi_n^0=(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2\left(x_n\right),\\
0\le n\le M, \left(u_n^1-u_n^0\right)\\
-\frac{\tau^2}{12}\left(\frac{\left(u_{n+1}^1-u_{n+1}^0\right)-2\left(u_n^1-u_n^0\right)+\left(u_{n-1}^1-u_{n-1}^0\right)}{h^2}\right)\\
+\frac{\tau^4}{144h^4}\left[(u_{n+2}^1-u_{n+2}^0)-4\left(u_{n+1}^1-u_{n+1}^0\right)\right.\\
+6\left(u_n^1-u_n^0\right)-4\left(u_{n-1}^1-u_{n-1}^0\right)\\
\left.+(u_{n-2}^1-u_{n-2}^0)\right]=\varphi_n^N,\\
\varphi_n^N=\left(-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{6}+\frac{\tau^4}{24}-\frac{\tau^5}{24}\right)\sin^2\left(x_n\right)\\
+\left(\frac{\tau^3}{6}-\frac{\tau^4}{12}-\frac{7}{36}\tau^5-\frac{15}{144}\tau^6\right.\\
\left.-\frac{25}{432}\tau^7-\frac{5}{432}\tau^8\right)\cos 2x_n\\
+\tau(\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2\left(x_n\right),\\
2\le n\le M-2, 0\le k\le N,\\
u_1^0-u_0^0=\frac{h}{2}\left(\frac{45u_0^0-154u_0^1+214u_0^2-156u_0^3+61u_0^4-10u_0^5}{12\tau^2}\right)\\
-\frac{h^2}{64\tau^3}\left(17u_0^0-71u_0^1+118u_0^2\right.\\
\left.-98u_0^3+41u_0^4-7u_0^5\right)\\
u_1^N-u_0^N=\frac{h}{212\tau^2}\left(45u_0^N-154u_0^{N-1}+214u_0^{N-2}\right.\\
\left.-156u_0^{N-3}+61u_0^{N-4}-10u_0^{N-5}\right)\\
-\frac{h^2}{64\tau^3}\left(17u_0^N-71u_0^{N-1}+118u_0^{N-2}\right.\\
\left.-98_0^{N-3}+41u_0^{N-4}-7u_0^{N-5}\right), u_1^k-u_0^k\\
=\frac{h}{2}\left(\frac{u_0^{k+1}-2u_0^k+u_0^{k-1}}{\tau^2}-\varphi_0^k\right),\quad 1\le k\le N-1
\end{cases}
\tag{20}
$$

for the approximate solution of problem (8). By rearranging coefficients in the problem we have again the $(N+1)\times(N+1)$ linear system (13) with matrix coefficients

$$
A=\begin{bmatrix}
0 & 0 & 0 & \dots & 0 & 0 & 0\\
x & y & x & \dots & 0 & 0 & 0\\
0 & x & y & \ddots & 0 & 0 & 0\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \ddots & y & x & 0\\
0 & 0 & 0 & \dots & x & y & x\\
-r & r & 0 & \dots & 0 & 0 & 0
\end{bmatrix},
$$

$$B=\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0\\ v & w & v & \ldots & 0 & 0 & 0\\ 0 & v & w & \ddots & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ddots & w & v & 0\\ 0 & 0 & 0 & \ldots & v & w & v\\ s & -s & 0 & \ldots & 0 & 0 & 0 \end{bmatrix},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & \ldots 0 & \frac{-1}{8} & 0\ldots & 0 & \frac{-1}{8}\\ m & n & m & 0 & \ldots & 0 & 0 & 0\\ 0 & m & n & m & \ldots & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & \ddots & n & m & 0\\ 0 & 0 & 0 & \ldots & \ldots & m & n & m\\ -t & t & 0 & \ldots & \ldots & 0 & 0 & 0 \end{bmatrix},$$

$$D=B,\ E=A,$$

$$R=\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0\\ 0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 1 & 0\\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix},$$

and with entries

$$x=\frac{\tau^2}{144h^4}, y=-\frac{\tau^2}{72h^4}, v=-\frac{1}{12h^2}-\frac{\tau^2}{36h^4},$$

$$w=-\frac{5}{6h^2}+\frac{\tau^2}{18h^4},$$

$$m=\frac{1}{\tau^2}+\frac{1}{6h^2}+\frac{\tau^2}{24h^4},$$

$$n=-\frac{2}{\tau^2}+\frac{5}{3h^2}-\frac{\tau^2}{12h^4},$$

$$r=\frac{\tau^4}{144h^4}, s=\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2},$$

$$t=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}.$$

Here U_s^k and φ_n^k are defined as

$$U_s^k=\begin{bmatrix} u_s^0\\ u_s^1\\ \vdots\\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

$$0\le k\le N, s=n-2, n-1, n, n+1, n+2.$$

$$\varphi_n^k=\begin{bmatrix} \varphi_n^0\\ \varphi_n^1\\ \cdot\\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, 0\le k\le N,$$

$$\varphi_n^0=(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2(x_n), 0\le n\le M,$$

$$\varphi_n^N=\left(-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{6}+\frac{\tau^4}{24}+\frac{\tau^5}{24}\right)\sin^2(x_n)$$

$$\varphi_n^k=\Big\{(\frac{5}{6}+\frac{\tau^2}{72})e^{-t_k}$$
$$+(\frac{1}{12}-\frac{\tau^2}{144})(e^{-t_{k+1}}+e^{-t_{k-1}})\Big\}\sin^2(x_n)$$
$$+\Big\{(-\frac{5}{3}+\frac{\tau^2}{9})e^{-t_k}$$
$$-(\frac{1}{6}+\frac{\tau^2}{18})(e^{-t_{k+1}}+e^{-t_{k-1}})\Big\}\cos 2x_n$$
$$+\Big\{\frac{\tau^3}{6}-\frac{\tau^4}{12}-\frac{7}{36}\tau^5-\frac{15}{144}\tau^6$$
$$-\frac{25}{432}\tau^7-\frac{5}{432}\tau^8\Big\}\cos 2x_n$$
$$+(\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2(x_n).$$

In exactly the same manner as Example 1 the linear system for the fourth order of accuracy difference scheme is solved with the following new formulas

$$U_M=-[P+Q(4I-\alpha_{M-1})^{-1}(\beta_{M-1}+3I)]^{-1}$$
$$\times\left[R+Q(4I-\alpha_{M-1})^{-1}\gamma_{M-1}\right], \quad (21)$$

$$U_{M-1}=[(\beta_{M-2}+5I)-(4I-\alpha_{M-2})\alpha_{M-1}]^{-1}$$
$$\times[(4I-\alpha_{M-2})\gamma_{M-1}-\gamma_{M-2}] \quad (22)$$

$$U_{M-2}=(4I-\alpha_{M-2})^{-1} \quad (23)$$
$$\times\{(5I+\beta_{M-2})U_{M-1}+\gamma_{M-2}\}$$

where

$$P=\frac{1}{12h}[25I+36\beta_{M-1}-16\alpha_{M-2}\beta_{M-1}$$
$$+3(\alpha_{M-3}\alpha_{M-2}\beta_{M-1}+\beta_{M-3}\beta_{M-1})],$$

$$Q=\frac{1}{12h}\left[-48I+36\alpha_{M-1}\right.$$

$$-16(\alpha_{M-2}\alpha_{M-1}+\beta_{M-1})$$

$$\left.+3(\alpha_{M-3}\alpha_{M-2}\alpha_{M-1}+\alpha_{M-3}\beta_{M-2}+\alpha_{M-1}\beta_{M-3})\right],$$

$$R=\frac{1}{12h}\left[36\gamma_{M-1}-16(\alpha_{M-2}\gamma_{M-1}+\gamma_{M-2})\right.$$

$$+3\left(\alpha_{M-3}\alpha_{M-2}\gamma_{M-1}\right.$$

$$\left.\left.+\alpha_{M-3}\gamma_{M-2}+\beta_{M-3}\gamma_{M-1}+\gamma_{M-3})\right)\right].$$

For the boundary condition $u_x(t,0)=0$, the system (18) with the matrix

$$T=\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & 0 & \ldots & 0\\ a & b & a & 0 & \ldots & \ldots & \ldots & \ldots & 0\\ 0 & a & b & a & 0 & \ldots & \ldots & \ldots & 0\\ 0 & 0 & a & b & a & 0 & \ldots & \ldots & \vdots\\ \vdots & \ldots & \ddots & \ddots & \ddots & \ddots & \ldots & \ldots & \vdots\\ 0 & \ldots & \ldots & \ldots & \ldots & a & b & a & 0\\ 0 & \ldots & 0 & \lambda_6 & \lambda_5 & \lambda_4 & \lambda_3 & \lambda_2 & \lambda_1 \end{bmatrix} \quad (24)$$

and the new entries

$$\lambda_1=\left(-\frac{1}{h}-\frac{45h}{24\tau^2}+\frac{17h^2}{24\tau^3}\right),$$

$$\lambda_2=\left(\frac{154h}{24\tau^2}-\frac{71h^2}{24\tau^3}\right),$$

$$\lambda_3=\left(-\frac{214h}{24\tau^2}+\frac{118h^2}{24\tau^3}\right),$$

$$\lambda_4=\left(\frac{156h}{24\tau^2}-\frac{98h^2}{12\tau^3}\right),$$

$$\lambda_5=\left(-\frac{61h}{24\tau^2}+\frac{41h^2}{24\tau^3}\right),\ \lambda_6=(\frac{10h}{24\tau^2}-\frac{7h^2}{24\tau^3}),$$

$$a=-\frac{h}{2\tau^2},\quad b=\left(-\frac{1}{h}+\frac{h}{\tau^2}\right)$$

is considered.

Example 2. *Consider*

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial^2 u(t,x)}{\partial x^2}=e^{-t}(\sin^2 x-2\cos 2x),\\ 0<t<1, 0<x<\pi,\\ u(0,x)=\frac{1}{10}u(1,x)+\frac{1}{10}u(\frac{1}{2},x)\\ +(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x, 0\le x\le\pi,\\ u_t(0,x)=\frac{1}{10}u_t(1,x)+\frac{1}{10}u_t(\frac{1}{2},x)\\ +(-1+\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x, 0\le x\le\pi,\\ u(t,0)=u_x(t,\pi)=0,\ 0\le t\le 1 \end{cases} \quad (25)$$

for one dimensional hyperbolic equation.

Note that this problem is similar to Example 1, with different mixed boundary conditions. Again exact solution of the problem is

$$u(t,x)=e^{-t}\sin^2 x.$$

In finding the approximate solution of problem (25), the method of first example is applied. Third and fourth orders of accuracy difference schemes (4), (6) are used. Approximating the boundary condition $u_x(t,\pi)=0$ the following formulas

$$U_M=-[P+Q(4I-\alpha_{M-1})^{-1}(\beta_{M-1}+3)]^{-1}$$

$$\times\{R+Q(4-\alpha_{M-1})^{-1}\gamma_{M-1}\}$$

$$U_{M-1}=-(P+Q)^{-1}R$$

$$U_{M-2}=(4I-\alpha_{M-2})^{-1}\{(5I+\beta_{M-2})U_{M-1}+\gamma_{M-2}\}$$

where

$$P=\frac{1}{6h}(11I+9\beta_{M-1}-2\alpha_{M-2}\beta_{M-1})U_M,$$

$$Q=\frac{1}{6h}\left[-18I+9\alpha_{M-1}\right.$$

$$\left.-2\left(\alpha_{M-2}\alpha_{M-1}+\beta_{M-2}\right)\right]U_{M-1}$$

$$R=\frac{1}{6h}(9\gamma_{M-1}-2\alpha_{M-2}\gamma_{M-1}-2\gamma_{M-2}),$$

for the third order of accuracy difference scheme and

$$U_M=-[P+Q(4I-\alpha_{M-1})^{-1}(\beta_{M-1}+3I)]^{-1}$$

$$\times\{R+Q(4I-\alpha_{M-1})^{-1}\gamma_{M-1}\},$$

$$U_{M-1}=[(\beta_{M-2}+5I)-(4I-\alpha_{M-2})\alpha_{M-1}]^{-1}$$

$$\times[(4I-\alpha_{M-2})\gamma_{M-1}-\gamma_{M-2}]$$

$$U_{M-2}=(4I-\alpha_{M-2})^{-1}\left\{(5I+\beta_{M-2})U_{M-1}+\gamma_{M-2}\right\},$$

where

$$P=\frac{1}{12h}\left[25I+36\beta_{M-1}-16\alpha_{M-2}\beta_{M-1}\right.$$

$$\left.+3(\alpha_{M-3}\alpha_{M-2}\beta_{M-1}+\beta_{M-3}\beta_{M-1})\right],$$

$$Q=\frac{1}{12h}\left[-48I+36\alpha_{M-1}\right.$$

$$-16(\alpha_{M-2}\alpha_{M-1}+\beta_{M-1})+3\left(\alpha_{M-3}\alpha_{M-2}\alpha_{M-1}\right.$$

$$\left.\left.+\alpha_{M-3}\beta_{M-2}+\alpha_{M-1}\beta_{M-3}\right)\right],$$

$$R=\frac{1}{12h}\left[36\gamma_{M-1}-16(\alpha_{M-2}\gamma_{M-1}+\gamma_{M-2})\right.$$

$$+3((\alpha_{M-3}\alpha_{M-2}\gamma_{M-1}$$

$$\left.+\alpha_{M-3}\gamma_{M-2}+\beta_{M-3}\gamma_{M-1}+\gamma_{M-3})\right]$$

for the fourth order of accuracy difference scheme are used. For the boundary condition $u(t,0)=0$ the following initial matrices

$$\alpha_1=\begin{bmatrix}0&0&\dots&0\\0&0&\dots&0\\\vdots&\vdots&\dots&\vdots\\0&0&\dots&0\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_1=\begin{bmatrix}0&0&\dots&0\\0&0&\dots&0\\\vdots&\vdots&\dots&\vdots\\0&0&\dots&0\end{bmatrix}_{(N+1)\times(N+1)}$$

$$\gamma_1=\gamma_2=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}_{(N+1)\times1},$$

$$\alpha_2=\begin{bmatrix}4/5&0&\dots&0\\0&4/5&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&4/5\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_2=\begin{bmatrix}-1/5&0&\dots&0\\0&-1/5&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&-1/5\end{bmatrix}_{(N+1)\times(N+1)}$$

are used in the formulae which were presented in (14).

Example 3. *Consider the NBVP with mixed condition*

$$\begin{cases}\frac{\partial^2u(t,x)}{\partial t^2}-\frac{\partial^2u(t,x)}{\partial x^2}=e^{-t}(\sin^2x-2\cos2x),\\0<t<1,0<x<\pi,\\u(0,x)=\frac{1}{10}u(1,x)+\frac{1}{10}u(\frac{1}{2},x)\\+(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2x,0\le x\le\pi,\\u_t(0,x)=\frac{1}{10}u_t(1,x)+\frac{1}{10}u_t(\frac{1}{2},x)\\+(-1+\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2x,0\le x\le\pi,\\u_x(t,0)=u(t,\pi)=0,\ \ 0\le t\le1\end{cases}\tag{26}$$

for one dimensional hyperbolic equation.

Note that this problem is similar to problem of Example1 with different boundary conditions. Exact solution of this problem is

$$u\left(t,x\right)=e^{-t}\sin^2x.$$

The approximate solution of problem (26) is obtained by a similar procedure as in the first example. Third and fourth order of accuracy difference schemes (4), (6) are used and the system

$$U_0=\alpha_1U_1+\beta_1U_2+\gamma_1$$

with

$$\alpha_1=\frac{-1}{h}T^{-1},\beta_1=0,\gamma_1=\frac{h}{2}T^{-1}\varphi_n^0$$

is considered. Matrices $T,\lambda_i,i=1,...,6;a,b$ are defined by (19) and (24) and are considered for the boundary condition $u_x(t,0)=0$. Approximating boundary condition $u(t,\pi)=0$, the following formulas

$$\begin{cases}U_{M-2}=\alpha_{M-1}U_{M-1}+\gamma_{M-1},\\U_{M-3}=\alpha_{M-2}U_{M-2}+\beta_{M-2}U_{M-1}+\gamma_{M-2},\\U_{M-3}=4U_{M-2}-5U_{M-1},\end{cases}$$

and

$$U_{M-1}=\left[(\beta_{M-2}+5I)-(4I-\alpha_{M-2})\alpha_{M-1}\right]^{-1}$$

$$\times[(4I-\alpha_{M-2})\gamma_{M-1}-\gamma_{M-2}]$$

are used.

The errors for the approximations are computed by the formula

$$E_M^N=\max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u\left(t_k,x_n\right)-U_n^k\right|^2h\right)^{\frac{1}{2}}.$$

Table 1. Error analysis for the approximate solutions of (8).

	N=20, M=400	N=30, M=900	N=40, M=1600
Third order of accuracy difference scheme	0,000211	0,00006181	0,00002605
Fourth order of accuracy difference scheme	0,00009415	0,00001866	0,000005752

Table 2. CPU times for the approximate solutions of (8).

	N=20, M=400	N=30, M=900	N=40, M=1600
Third order of accuracy difference scheme	2.3078	13.5915	68.6596
Fourth order of accuracy difference scheme	2.3473	13.5283	67.7495

Table 3. Error analysis for the approximate solutions of (25).

	N=20, M=400	N=30, M=900	N=40, M=1600
Third order of accuracy difference scheme	0,0004817	0,0002047	0,0001138
Fourth order of accuracy difference scheme	0,00009781	0,00001979	0,00001954

Table 4. CPU times for the approximate solutions of (25).

	N=20, M=400	N=30, M=900	N=40, M=1600
Third order of accuracy difference scheme	2.3031	13.5165	68.2402
Fourth order of accuracy difference scheme	1.7361	13.5241	68.8427

Table 5. Error analysis for the approximate solutions of (26).

	N=20, M=400	N=30, M=900	N=40, M=1600
Third order of accuracy difference scheme	0,0037	0,0011	0,0004602
Fourth order of accuracy difference scheme	0,00009415	0,00001866	0,000005752

Table 6. CPU times for the approximate solutions of (26).

	N=20, M=400	N=30, M=900	N=40, M=1600
Third order of accuracy difference scheme	1.7006	13.4295	68.1728
Fourth order of accuracy difference scheme	1.7401	13.4628	68.0591

Here $u(t_k, x_n)$ represents exact solution and U_n^k represents numerical solution at (t_k, x_n). We denote the third order of accuracy difference scheme (4) as TO and the fourth order of accuracy difference scheme (6) as FO. Errors and the related CPU times are represented in Table 1,3,5 and Table 2,4,6 respectively, for different M and N values. The implementations are carried out by Matlab 7.9.0 software package and obtained by a PC System 64bit, Intel R Core TM i5 CPU, 3.20 GHz, 3.60Hz, 4000Mb of RAM.

The following conclusions can be noted from the tables above for the comparison of the numerical results presented in the tables.

- From Table 1 and Table 2, it can be noticed that approximately the same accuracy is achieved by TO with data error ,N=40, M=1600 and by FO with data error N=30, M=900 in different CPU times; 68.6596s and 13.5283s, respectively. This means the use of the difference scheme FO accelerates the computation with a ratio of more than 68.66/13.5≈5.08 times, that is, FO is considerably faster than TO.
- In Table 3 and Table 4, almost the same accuracy is achieved by TO with error ,N=40, M=1600 and by FO with error N=20, M=400 in different CPU

times; 68.6596s and 13.5283s, respectively, which means that the use of the difference scheme FO accelerates the computation with a ratio of more than 68.24/1.73≈39.44 times, which shows that FO is faster than TO.

- In Table 5 and Table 6, it is noted that approximately similar accuracy is achieved by TO with data error ,N=40, M=1600 and by FO with data error N=20, M=400 in different CPU times; 68.1728s and 1.7401s, respectively. This means that the use of the difference scheme FO accelerates the computation with a ratio of more than 68.17/1.74≈39.17 times, that is, FO is approximately faster than TO.
- It can be concluded from the tables that numerical results become approximately the same for larger N and M values for each difference scheme in the reliable range of the CPU times and this shows that the approximate solutions of problem (8), (25), (26) are accurate.
- In conclusion, the fourth order of accuracy difference scheme is more accurate than the third order of accuracy difference scheme when considering the CPU times and the error levels.

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RESEARCH ARTICLE

Fractional Hermite-Hadamard type inequalities for functions whose derivatives are extended s-(α, m)-preinvex

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ABSTRACT

In this paper, we introduce the class of extended s-(α, m)-preinvex functions. We establish a new fractional integral identity and derive some new fractional Hermite-Hadamard type inequalities for functions whose derivatives are in this novel class of function.

1. Introduction

It is well known that convexity plays an important and central role in many areas, such as economic, finance, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions.

One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard integral inequality

$$f\left(\tfrac{a+b}{2}\right) \le \tfrac{1}{b-a}\int_a^b f(x)dx \le \tfrac{f(a)+f(b)}{2}, \qquad (1)$$

where f is a real continuous convex function on the finite interval $[a, b]$. If the function f is concave, then (1) holds in the reverse direction (see [1]).

The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants have appeared in the literature, see [2–9] and references cited therein.

Kirmaci et al. [10] presented some results connected with inequality (1)

$$\left| \tfrac{1}{b-a}\int_a^b f(x)dx - f\left(\tfrac{a+b}{2}\right) \right| \le \tfrac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Recently, Sarikaya et al [11], gave the fractional analogue of (1)

$$f\left(\tfrac{a+b}{2}\right) \le \tfrac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[\left(J_{a^+}^\alpha f\right)(b) + \left(J_{b^-}^\alpha f\right)(a)\right] \le \tfrac{f(a)+f(b)}{2}. \qquad (2)$$

Zhu et al [12] established the following result connected with inequality (2).

$$\left| \tfrac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[\left(J_{a^+}^\alpha f\right)(b) + \left(J_{b^-}^\alpha f\right)(a)\right] - f\left(\tfrac{a+b}{2}\right) \right| \le \tfrac{b-a}{4(1+\alpha)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)\left(\alpha + 3 - \frac{1}{2^{\alpha-1}}\right).$$

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Motivated by the above results, in this paper, we introduce the class of extended s-(α, m)-preinvex functions. We establish a new fractional integral identity and derive some new fractional Hermite-Hadamard type inequalities for functions whose derivatives are in this novel class of functions.

2. Preliminaries

In this section we recall some definitions and lemmas

Definition 1. *[13] A function $f : I \to \mathbb{R}$ is said to be convex, if*

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2. *[14] A nonnegative function $f : I \to \mathbb{R}$ is said to be P-convex, if*

$$f\left(tx + (1-t)\,y\right) \leq f\left(x\right) + f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 3. *[15] A nonnegative function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function, if*

$$f\left(tx + (1-t)\,y\right) \leq \frac{f\left(x\right)}{t} + \frac{f(y)}{1-t}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 4. *[16] A nonnegative function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin function, where $s \in [0,1]$, if*

$$f\left(tx + (1-t)\,y\right) \leq \frac{f\left(x\right)}{t^s} + \frac{f(y)}{(1-t)^s}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 5. *[17] A nonnegative function $f : I \to \mathbb{R}$ is said to be α-Godunova-Levin function, where $\alpha \in (0,1]$, if*

$$f\left(tx + (1-t)\,y\right) \leq \frac{f\left(x\right)}{t^\alpha} + \frac{f(y)}{1-t^\alpha}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 6. *[18] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be α-convex in the first sense for some fixed $\alpha \in (0,1]$, if*

$$f(tx + (1-t)y) \ \leq t^\alpha f(x) + (1-t^\alpha)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 7. *[19] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if*

$$f(tx + (1-t)y) \ \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 8. *[20] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be extended s-convex for some fixed $s \in [-1,1]$, if*

$$f(tx + (1-t)y) \ \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Definition 9. *[21] A function $f : [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in (0,1]$,if*

$$f\left(tx + m\,(1-t)\,y\right) \leq tf\left(x\right) + m\,(1-t)\,f(y)$$

holds for all $x, y \in I$, and $t \in [0,1]$.

Definition 10. *[22] A function $f : [0,b] \to \mathbb{R}$ is said to be (α, m)-convex, where $\alpha, m \in (0,1]$,if*

$$f\left(tx + m\,(1-t)\,y\right) \leq t^\alpha f\left(x\right) + m\left(1-t^\alpha\right) f(y)$$

holds for all $x, y \in I$, and $t \in [0,1]$.

Definition 11. *[23] A function $f : [0,b] \to \mathbb{R}$ is said to be (s,m)-convex, where $\alpha, m \in (0,1]$,if*

$$f\left(tx + m\,(1-t)\,y\right) \leq t^s f\left(x\right) + m\,(1-t)^s\, f(y)$$

holds for all $x, y \in I$, and $t \in [0,1]$.

Definition 12. *[24] A function $f : I \to \mathbb{R}$ is said to be (α, m)-Godunova-Levin functions of first kind, where $\alpha, m \in (0,1]$, if*

$$f\left(tx + m\,(1-t)\,y\right) \leq \frac{f\left(x\right)}{t^\alpha} + m\frac{f(y)}{1-t^\alpha}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 13. *[24] A function $f : I \to \mathbb{R}$ is said to be (s,m)-Godunova-Levin functions of first kind, where $s \in [0,1]$ and $m \in (0,1]$, if*

$$f\left(tx + m\,(1-t)\,y\right) \leq \frac{f\left(x\right)}{t^s} + m\frac{f(y)}{(1-t)^s}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 14. *[25] A nonnegative function $f : I \subset [0, \infty) \to [0, \infty)$ is said to be s-(α, m)-convex in the second sense where $\alpha, m \in [0, 1]$ and $s \in (0, 1]$, if the following inequality*

$$f(tx + (1-t)y) \leq (1-t^{\alpha})^s f(x) + m(t^{\alpha})^s f(\frac{y}{m})$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 15. *[26] A set $K \subseteq \mathbb{R}^n$ is said an invex with respect to the bifunction $\eta : K \times K \to \mathbb{R}^n$, if for all $x, y \in K$, we have*

$$x + t\eta(y, x) \in K.$$

In what follows we assume that $K \subseteq \mathbb{R}$ be an invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$.

Definition 16. *[26] A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if*

$$f(x + t\eta(y, x)) \leq (1-t) f(x) + tf(y)$$

holds for all $x, y \in K$ and all $t \in [0, 1]$.

Definition 17. *[27] A nonnegative function $f : K \to \mathbb{R}$ is said to be P-preinvex function with respect to η, if*

$$f(x + t\eta(y, x)) \leq f(x) + f(y)$$

holds for all $x, y \in K$ and all $t \in [0, 1]$.

Definition 18. *[27] A nonnegative function $f : K \to \mathbb{R}$ is said to be Godunova-Levin preinvex function with respect to η, if*

$$f(x + t\eta(y, x)) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}$$

holds for all $x, y \in K$ and all $t \in (0, 1)$.

Definition 19. *[28] A nonnegative function $f : K \to \mathbb{R}$ is said to be s-Godunova-Levin preinvex function with respect to η, where $s \in [0, 1]$, if*

$$f(x + t\eta(y, x)) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}$$

holds for all $x, y \in K$ and all $t \in (0, 1)$.

Definition 20. *[29] A nonnegative function $f : K \subset [0, \infty) \to \mathbb{R}$ is said to be α-preivex in the first sense with respect to η for some fixed $\alpha \in (0, 1]$, if*

$$f(x + t\eta(y, x)) \leq (1-t^{\alpha}) f(x) + t^{\alpha} f(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 21. *[30] A nonnegative function $f : K \subset [0, \infty) \to \mathbb{R}$ is said to be s-preinvex in the second sense with respect to η for some fixed $s \in (0, 1]$, if*

$$f(x + t\eta(y, x)) \leq (1-t)^s f(x) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 22. *[31] A function $f : K \subset [0, b^*] \to \mathbb{R}$ is said to be m-preinvex with respect to η where $b^* > 0$ and $m \in (0, 1]$, if*

$$f(x + t\eta(y, x)) \leq (1-t) f(x) + mtf(\frac{y}{m})$$

holds for all $x, y \in K$, and $t \in [0, 1]$.

Definition 23. *[31] A function $f : K \to \mathbb{R}$ is said to be (α, m)-preinvex with respect to η for some fixed $\alpha \in (0, 1]$, and $m \in (0, 1]$, if*

$$f(x + t\eta(y, x)) \leq (1-t^{\alpha}) f(x) + mt^{\alpha} f(\frac{y}{m})$$

holds for all $x, y \in K$, and $t \in [0, 1]$.

Definition 24. *[32] A function $f : K \subset [0, b^*] \to \mathbb{R}$ is said to be (s, m)-preinvex with respect to η for some fixed $\alpha \in (0, 1]$where $b^* > 0$ and $m \in (0, 1]$, if*

$$f(x + t\eta(y, x)) \leq (1-t)^s f(x) + mt^s f(\frac{y}{m})$$

holds for all $x, y \in K$, and $t \in [0, 1]$.

Lemma 1. *[33] For $t, n \in [0, 1]$, we have*

$$(1-t)^n \leq 2^{1-n} - t^n.$$

Lemma 2. *[34] For any $0 \leq a < b$ and fixed $p \geq 1$, we have*

$$(b-a)^p \leq b^p - a^p.$$

We also recall that the incomplete beta function is defined as follows:

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dx$$

for $x \in [0,1]$ and $\alpha,\beta > 0$, where $B_1(\alpha,\beta) = B(\alpha,\beta)$ is the beta function.

3. Main results

In what follows we assume that $[a, a+\eta(b,a)] \subset K \subset [0, b^*]$ where $b^* > 0$ such that K is an invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$.

Definition 25. *A nonnegative function $f : K \to [0,\infty)$ is said to be extended s-(α,m)-preinvex in the second sense where $\alpha, m \in (0,1]$ and $s \in [-1,1]$, if the following inequality*

$$f(x + t\eta(y,x)) \leq (1-t^\alpha)^s f(x) + m(t^\alpha)^s f(\frac{y}{m})$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Remark 1. *Definition 25 includes all the definitions cited above, except for Definition 15.*

Lemma 3. *Let $f : [a, a+\eta(b,a)] \to \mathbb{R}$ be a differentiable mapping on $(a, a+\eta(b,a))$ with $\eta(b,a) > 0$, and assume that $f' \in L([a, a+\eta(b,a)])$, then the following equality holds*

$$\begin{aligned} &\frac{\Gamma(\delta+1)}{2\eta^\delta(b,a)}\left[\left(J^\delta_{a^+}f\right)(a+\eta(b,a)) + \left(J^\delta_{(a+\eta(b,a))^-}f\right)(a)\right] - f\left(\tfrac{2a+\eta(b,a)}{2}\right) \quad (3)\\ =& \frac{\eta(b,a)}{2}\left(\int_0^1 kf'(a+t\eta(b,a))\,dt - \int_0^1\left(t^\delta-(1-t)^\delta\right)f'(a+t\eta(b,a))\,dt\right), \end{aligned}$$

where

$$k = \begin{cases} 1 \text{ if } 0 \leq t < \frac{1}{2}, \\ -1 \text{ if } \frac{1}{2} \leq t < 1. \end{cases} \quad (4)$$

Proof. Let

$$\begin{aligned} I &= \int_0^1 kf'(a+t\eta(b,a))\,dt - \int_0^1\left(t^\delta-(1-t)^\delta\right)f'(a+t\eta(b,a))\,dt \\ &= I_1 - I_2, \end{aligned} \quad (5)$$

where

$$I_1 = \int_0^1 kf'(a+t\eta(b,a))\,dt, \quad (6)$$

and

$$I_2 = \int_0^1\left(t^\delta-(1-t)^\delta\right)f'(a+t\eta(b,a))\,dt, \quad (7)$$

k is defined by (3).
Clearly,

$$\begin{aligned} I_1 &= \frac{2}{\eta(b,a)}\left[f\left(\tfrac{2a+\eta(b,a)}{2}\right) - (f(a) + f(a+\eta(b,a)))\right]. \end{aligned} \quad (8)$$

Now, by integration by parts, I_2 gives

$$\begin{aligned} I_2 =& \tfrac{1}{\eta(b,a)}f(a+\eta(b,a)) + \tfrac{1}{\eta(b,a)}f(a) \\ &- \tfrac{\delta}{\eta(b,a)}\left(\int_0^1 t^{\delta-1}f(a+t\eta(b,a))\,dt + \int_0^1(1-t)^{\delta-1}f(a+t\eta(b,a))\,dt\right) \\ =& \tfrac{1}{\eta(b,a)}f(a+\eta(b,a)) + \tfrac{1}{\eta(b,a)}f(a) \\ &- \tfrac{\alpha}{\eta^{\delta+1}(b,a)}\left(\int_a^{a+\eta(a,b)}(u-a)^{\delta-1}f(u)\,du + \int_a^{a+\eta(a,b)}(\eta(b,a)+a-u)^{\delta-1}f(u)\,du\right) \\ =& \tfrac{1}{\eta(b,a)}f(a+\eta(b,a)) + \tfrac{1}{\eta(b,a)}f(a) \\ &- \tfrac{\Gamma(\delta+1)}{\eta^{\delta+1}(b,a)}\left(\left(I^\delta_{a^+}f\right)(a+\eta(b,a)) + \left(I^\delta_{(a+\eta(b,a))^-}f\right)(a)\right). \end{aligned} \quad (9)$$

Combining (8), (9) and (5), we obtain the desired equality in (3). □

Theorem 1. *Let $f : [a, a+\eta(b,a)] \to \mathbb{R}$ be a positive differentiable mapping on $(a, a+\eta(b,a))$ with $\eta(b,a) > 0$ and $f' \in L([a, a+\eta(b,a)])$. If $|f'|$ is extended s-(α,m)-preinvex function where $\alpha, m \in (0,1]$ and $s \in (-1,1]$, then the following fractional inequality holds for $\alpha s + \delta \neq -1$*

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J_{a^+}^{\delta}f\right)(a+\eta(b,a))+\left(J_{(a+\eta(b,a))^-}^{\delta}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(2^{1-s}-\tfrac{1}{\alpha s+1}+\tfrac{2^{2-s}}{\delta+1}\left(1-\left(\tfrac{1}{2}\right)^{\delta}\right)-\tfrac{1}{\alpha s+\delta+1}-B(\alpha s+1,\delta+1)\right)|f'(a)|$$
$$+m\left(\tfrac{1}{\alpha s+1}+2B_{\frac{1}{2}}(\alpha s+1,\delta+1)-B(\alpha s+1,\delta+1)+\tfrac{1}{\alpha s+\delta+1}\left(1-\tfrac{1}{2^{\alpha s+\delta}}\right)\right)\left|f'\left(\tfrac{b}{m}\right)\right|,$$

where $B(.,.)$ and $B_{\frac{1}{2}}(.,.)$ are the beta and the incomplete beta functions respectively.

Proof. From Lemma 3, and properties of modulus we have

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J_{a^+}^{\delta}f\right)(a+\eta(b,a))+\left(J_{(a+\eta(b,a))^-}^{\delta}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1 |f'(ta+(1-t)b)|\,dt+\int_0^{\frac{1}{2}}\left((1-t)^{\delta}-t^{\delta}\right)|f'(ta+(1-t)b)|\,dt+\int_{\frac{1}{2}}^1\left(t^{\delta}-(1-t)^{\delta}\right)|f'(ta+(1-t)b)|\,dt\right). \tag{10}$$

Since $|f'|$ is extended s-(α,m)-preinvex function, (10) gives

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J_{a^+}^{\delta}f\right)(a+\eta(b,a))+\left(J_{(a+\eta(b,a))^-}^{\delta}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1 (1-t^{\alpha})^s\times|f'(a)|+m(t^{\alpha})^s\left|f'(\frac{b}{m})\right|dt\right.$$
$$+\int_0^{\frac{1}{2}}\left((1-t)^{\delta}-t^{\delta}\right)\left((1-t^{\alpha})^s|f'(a)|+mt^{\alpha s}\left|f'(\frac{b}{m})\right|\right)dt$$
$$\left.+\int_{\frac{1}{2}}^1\left(t^{\delta}-(1-t)^{\delta}\right)\left((1-t^{\alpha})^s|f'(a)|+mt^{\alpha s}\left|f'(\frac{b}{m})\right|\right)dt\right). \tag{11}$$

Now, applying Lemma 1 for (11), we get

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J_{a^+}^{\delta}f\right)(a+\eta(b,a))+\left(J_{(a+\eta(b,a))^-}^{\delta}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\left(\left(\int_0^1\left(2^{1-s}-t^{\alpha s}\right)dt+\int_0^{\frac{1}{2}}\left(2^{1-s}\left((1-t)^{\delta}-t^{\delta}\right)\times\left(t^{\alpha s+\delta}-t^{\alpha s}(1-t)^{\delta}\right)\right)dt\right.\right.\right.$$
$$\left.\left.+\int_{\frac{1}{2}}^1\left(2^{1-s}\left(t^{\delta}-(1-t)^{\delta}\right)-t^{\alpha s+\delta}-t^{\alpha s}(1-t)^{\delta}\right)dt\right)|f'(a)|$$
$$\left.+m\left(\int_0^{\frac{1}{2}}\left(t^{\alpha s}(1-t)^{\delta}-t^{\alpha s+\delta}\right)dt+\int_{\frac{1}{2}}^1\left(t^{\alpha s+\delta}-t^{\alpha s}(1-t)^{\delta}\right)dt+\int_0^1 t^{\alpha s}dt\right)\right)\left|f'\left(\tfrac{b}{m}\right)\right|$$
$$=\frac{\eta(b,a)}{2}\left(\left(2^{1-s}-\tfrac{1}{\alpha s+1}+\tfrac{2^{2-s}}{\delta+1}\left(1-\left(\tfrac{1}{2}\right)^{\delta}\right)-\tfrac{1}{\alpha s+\delta+1}-B(\alpha s+1,\delta+1)\right)|f'(a)|\right.$$
$$\left.+m\left(\tfrac{1}{\alpha s+1}+2B_{\frac{1}{2}}(\alpha s+1,\delta+1)-B(\alpha s+1,\delta+1)\times\tfrac{1}{\alpha s+\delta+1}\left(1-\tfrac{1}{2^{\alpha s+\delta}}\right)\right)\left|f'\left(\tfrac{b}{m}\right)\right|\right),$$

which is the desired result. □

Remark 2. *Theorem 1 will be reduces to Theorem 2.3 from [12], if we choose $s=\alpha=m=1$ and $\eta(b,a)=b-a$.*

Theorem 2. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a positive differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ and $f'\in L([a,a+\eta(b,a)])$. If $|f'|^q$ $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, is extended s-(α,m)-preinvex function, where $\alpha,m\in(0,1]$ and $s\in[-1,1]$, and $q>1$, then the following fractional inequality holds for $s\alpha\neq-1$*

$$\begin{aligned}&\left|\tfrac{\Gamma(\delta+1)}{2\eta^{\alpha}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a))\right.\right.\\&\left.\left.+\left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|\\\leq\;&\tfrac{\eta(b,a)}{2}\left(\left((2^{1-s}-\tfrac{1}{s\alpha+1})\left|f'(a)\right|^q\right.\right.\\&\left.+\tfrac{m}{s\alpha+1}\left|f'\left(\tfrac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\\&+\left(\tfrac{1}{\delta p+1}\left(1-\left(\tfrac{1}{2}\right)^{\delta p}\right)\right)^{\frac{1}{p}}\\&\times\left(\left(\left(\tfrac{1}{2^s}-\tfrac{1}{(s\alpha+1)2^{s\alpha+1}}\right)\left|f'(a)\right|^q\right.\right.\\&\left.+\tfrac{m}{(s\alpha+1)2^{s\alpha+1}}\left|f'\left(\tfrac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\\&\times\left(\left(\tfrac{1}{2^s}-\tfrac{2^{s\alpha+1}-1}{(s\alpha+1)2^{s\alpha+1}}\right)\left|f'(a)\right|^q\right.\\&\left.\left.\left.+\;m\tfrac{2^{s\alpha+1}-1}{(s\alpha+1)2^{s\alpha+1}}\left|f'\left(\tfrac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Proof. From Lemma 3, properties of modulus, Hölder inequality, and Lemma 2, we have

$$\begin{aligned}&\left|\tfrac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a))\right.\right.\\&\left.\left.+\left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|\\\leq\;&\tfrac{\eta(b,a)}{2}\left(\left(\int_0^1dt\right)^{1-\frac{1}{q}}\right.\\&\times\left(\int_0^1\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\\&+\left(\int_0^{\frac{1}{2}}\left((1-t)^{\delta}-t^{\delta}\right)^pdt\right)^{\frac{1}{p}}\\&\times\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{2}}^1\left((1-t)^{\delta}-t^{\delta}\right)^pdt\right)^{\frac{1}{p}}\\&\left.\times\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\right)\\\leq\;&\tfrac{\eta(b,a)}{2}\left(\left(\int_0^1\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\right.\\&+\left(\int_0^{\frac{1}{2}}\left((1-t)^{\delta p}-t^{\delta p}\right)dt\right)^{\frac{1}{p}}\\&\times\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{2}}^1\left(t^{\delta p}-(1-t)^{\delta p}\right)dt\right)^{\frac{1}{p}}\\&\left.\times\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\right)\\=\;&\tfrac{\eta(b,a)}{2}\left(\left(\int_0^1\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\right.\\&+\left(\tfrac{1}{\delta p+1}\left(1-\tfrac{1}{2^{\delta p}}\right)\right)^{\frac{1}{p}}\\&\times\left(\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\right.\\&\left.\left.+\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^qdt\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Using the fact that $|f'|^q$ is extended s-preinvex function, and Lemma 1, (3) gives

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\alpha}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a)) + \left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right] - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$

$$\leq \frac{\eta(b,a)}{2}\left(\left(\int_0^1 (2^{1-s}-t^{s\alpha})\left|f'(a)\right|^q + m\left(t^{s\alpha}\right)\left|f'(\frac{b}{m})\right|^q dt\right)^{\frac{1}{q}} + \left(\frac{1}{\delta p+1}\left(1-\left(\frac{1}{2}\right)^{\delta p}\right)\right)^{\frac{1}{p}} \times\left(\left(\int_0^{\frac{1}{2}} (2^{1-s}-t^{s\alpha})\left|f'(a)\right|^q + m\left(t^{s\alpha}\right)\left|f'(\frac{b}{m})\right|^q dt\right)^{\frac{1}{q}} \times\left(\int_{\frac{1}{2}}^1 (2^{1-s}-t^{s\alpha})\left|f'(a)\right|^q + m\left(t^{s\alpha}\right)\left|f'(\frac{b}{m})\right|^q dt\right)^{\frac{1}{q}}\right)\right)$$

$$= \frac{\eta(b,a)}{2}\left(\left((2^{1-s}-\frac{1}{s\alpha+1})\left|f'(a)\right|^q + m\frac{1}{s\alpha+1}\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}} + \left(\frac{1}{\delta p+1}\left(1-\left(\frac{1}{2}\right)^{\delta p}\right)\right)^{\frac{1}{p}} \times\left(\left(\left(\frac{1}{2^s}-\frac{1}{(s\alpha+1)2^{s\alpha+1}}\right)\left|f'(a)\right|^q + m\left(\frac{1}{(s\alpha+1)2^{s\alpha+1}}\right)\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}} \times\left(\left(\frac{1}{2^s}-\frac{2^{s\alpha+1}-1}{(s\alpha+1)2^{s\alpha+1}}\right)\left|f'(a)\right|^q + m\left(\frac{2^{s\alpha+1}-1}{(s\alpha+1)2^{s\alpha+1}}\right)\left|f'(\frac{b}{m})\right|^q\right)^{\frac{1}{q}}\right)\right)$$

which is the desired result. □

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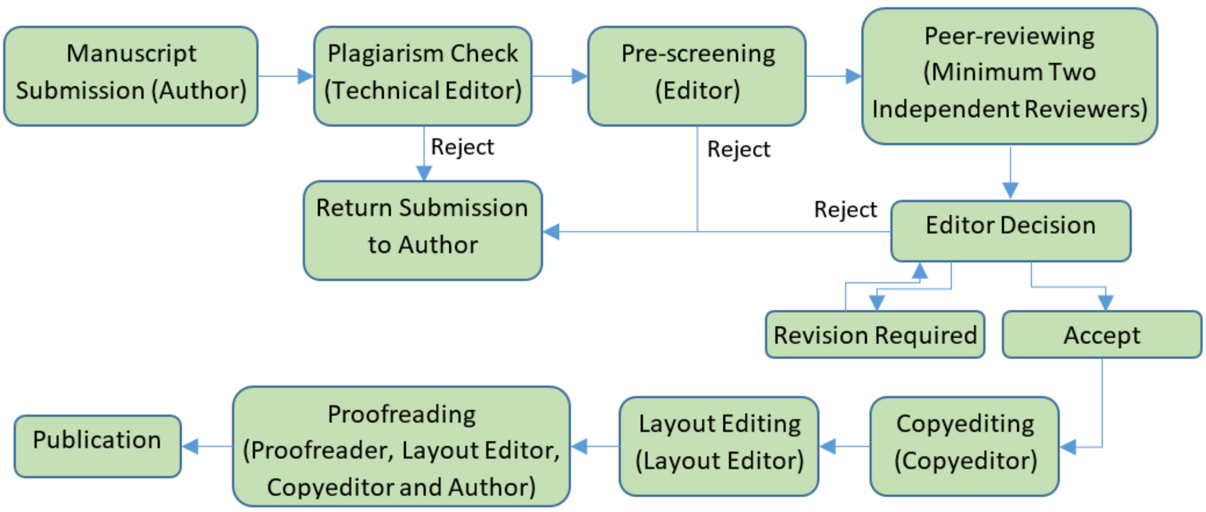

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