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RESEARCH ARTICLE

Using matrix stability for variable telegraph partial differential equation

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ABSTRACT

The variable telegraph partial differential equation depend on initial boundary value problem has been studied. The coefficient constant time-space telegraph partial differential equation is obtained from the variable telegraph partial differential equation throughout using Cauchy-Euler formula. The first and second order difference schemes were constructed for both of coefficient constant time-space and variable time-space telegraph partial differential equation. Matrix stability method is used to prove stability of difference schemes for the variable and coefficient telegraph partial differential equation. The variable telegraph partial differential equation and the constant coefficient time-space telegraph partial differential equation are compared with the exact solution. Finally, approximation solution has been found for both equations. The error analysis table presents the obtained numerical results.

1. Introduction

Partial differential equations have several applications in engineering, finance, physics and seismology [1–3]. They have several approximation methods which are different from each other. Some of these methods are solvable with respect to variables time and space. The space- heat equations were presented by difference schemes in previous works [4–6]. The partial differential equations depend on time were worked on in some papers [7–9], The telegraph partial differential equations is a special equation of the partial differential equations. In the literature, Telegraph equations can be defined based on time and space. Many important studies have been done on these equations in [10–12]. The telegraph partial differential equations were solved by difference schemes and methods in [13–16].

In this paper, the initial boundary value problem for variable coefficient partial differential equation is investigated

$$\begin{cases} \frac{\partial}{\partial t}\left(\alpha(t)u_t(t,x)\right) - \frac{\partial}{\partial x}\left(\beta(x)u_x(t,x)\right) + pu(t,x) \\ = f(t,x),\ 0<t<T,\ 0<x<L \\ u(0,x)=\varphi(x), \quad u_t(0,x)=\psi(x), \quad 0\le t\le T, \\ u(t,0)=g_1(t), \quad u(t,L)=g_2(t), \quad 0\le x\le L. \end{cases} \tag{1}$$

Here, $\alpha(t)$, $\beta(x)$ are variable as to t, x, respectively. Now, we shall construct first order difference scheme. Then, we will prove the stability estimates for this problem.

2. First and second order difference schemes for variable telegraph partial differential equation

If taking as $\alpha(t)=t^2$, $\beta(x)=x^2$ and $p=1$ in the formula (1), this formula can be written as follow

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$$\begin{cases} t^2u_{tt}(t,x)+2tu_t(t,x)-x^2u_{xx}(t,x)-2xu_x(t,x)\\ +u(t,x)=f(t,x),\ 1<t<e^T,\ 1<x<e^L\\ u(0,x)=In(\varphi(x)),u_t(0,x)=In(\psi(x)),\\ u(t,0)=u(t,L)=0,\quad 1\le t\le e^T,\ 1\le x\le e^L. \end{cases} \tag{2}$$

This equation represents a variable time-space telegraph partial differential equation. It is not easy to find out the analytical solution of this equation.

Therefore, if the Cauchy-Euler formula is applied to the last part of the equation separately for the x and t variables, the formula (2) can be written as

$$\begin{cases} u_{tt}(t,x)+u_t(t,x)-u_{xx}(t,x)-u_x(t,x)+u(t,x)\\ =f(t,x),\ 0<t<T,\ 0<x<L\\ u(0,x)=\varphi(x),\quad u_t(0,x)=\psi(x),\quad 0\le t\le T,\\ u(t,0)=u(t,L)=0,\quad 0\le x\le L. \end{cases} \tag{3}$$

The problem (3) is a coefficient time-space telegraph partial differential equation.

Now, we shall construct the first and the second order of accuracy difference scheme for the equation (2). In the first step, we consider the set $w_{\tau,h}=[0,1]_\tau\times[0,\pi]_h$ of a family of grid points depending on the small parameters τ and h. To evaluate difference scheme for problem (2), the following formula

$$[0,1]_\tau\times[0,\pi]_h=\{(t_k,x_n):t_k=k\tau,0\le k\le N,\\ N\tau=1,x_n=nh,0\le n\le M;Mh=\pi\},$$

is used. For the formula (2), we get the first order difference scheme

$$\begin{cases} t_k^2\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2t_k\dfrac{u_n^{k+1}-u_n^k}{\tau}\\ -x_n^2\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}-2x_n\dfrac{u_{n+1}^k-u_{n-1}^k}{2h}\\ +u_n^k=f_n^k,\quad x_n=nh,\ t_k=k\tau,\\ 1\le k\le N-1,\ 1\le n\le M-1,\\ u_0^k=u_M^k=0,\ u_n^0=In(\varphi(x_n)),\ 0\le k\le N\\ \dfrac{u_n^1-u_n^0}{\tau}=In(\psi(x_n)),\ 0\le n\le M, \end{cases} \tag{4}$$

and the second order difference scheme for the formula (2)

$$\begin{cases} t_k^2\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2t_k\dfrac{u_n^{k+1}-u_n^{k-1}}{2\tau}\\ -\frac{x_n^2}{2}\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}\\ -\frac{x_n^2}{2}\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\\ -\frac{x_n}{2}\dfrac{u_{n+1}^{k+1}-u_{n-1}^{k+1}}{h}-\frac{x_n}{2}\dfrac{u_{n+1}^{k-1}-u_{n-1}^{k-1}}{h}\\ +\frac12u_n^{k+1}+\frac12u_n^{k-1}=f_n^k,\\ x_n=nh,\ t_k=k\tau,\ 1\le k\le N-1,\ 1\le n\le M-1,\\ \dfrac{u_n^1-u_n^0}{\tau}=In(\psi(x_n))+\frac{\tau}{2}\dfrac{u_n^2-2u_n^1+u_n^0}{\tau^2},\\ u_n^0=In(\varphi(x_n)),u_0^k=u_M^k=0,\\ 0\le k\le N,\ 0\le n\le M. \end{cases} \tag{5}$$

Similarly, the first order difference schemes for the formula (3) are

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_n^{k+1}-u_n^k}{\tau}-\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\\ -\dfrac{u_{n+1}^k-u_{n-1}^k}{2h}+u_n^k=f_n^k,\quad x_n=nh,\ t_k=k\tau,\\ 1\le k\le N-1,\ 1\le n\le M-1,\\ u_0^k=u_M^k=0,\ u_n^0=\varphi(x_n),\ \dfrac{u_n^1-u_n^0}{\tau}=\psi(x_n),\\ 0\le k\le N,\ 0\le n\le M, \end{cases} \tag{6}$$

and the second order difference schemes

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_n^{k+1}-u_n^{k-1}}{2\tau}\\ -\frac12\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}\\ -\frac12\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\\ -\frac14\dfrac{u_{n+1}^{k+1}-u_{n-1}^{k+1}}{h}-\frac14\dfrac{u_{n+1}^{k-1}-u_{n-1}^{k-1}}{h}\\ +\frac12u_n^{k+1}+\frac12u_n^{k-1}=f_n^k,\\ x_n=nh,\ t_k=k\tau,\ 1\le k\le N-1,\ 1\le n\le M-1,\\ u_n^0=\varphi(x),\ \dfrac{u_n^1-u_n^0}{\tau}=\psi(x)+\frac{\tau}{2}\dfrac{u_n^2-2u_n^1+u_n^0}{\tau^2},\\ u_0^k=u_M^k=0,\quad 0\le k\le N,\ 0\le n\le M. \end{cases} \tag{7}$$

The formula (4) is rewritten as

$$\left(\frac{t_k^2}{\tau^2}+2\frac{t_k}{\tau}\right)u_n^{k+1}+\left(-\frac{x_n^2}{h^2}-\frac{x_k}{h}\right)u_{n+1}^k$$
$$+\left(-2\frac{t_k^2}{\tau^2}-2\frac{t_k}{\tau}+1+2\frac{x_n^2}{h^2}\right)u_n^k \quad (8)$$
$$+\left(-\frac{x_n^2}{h^2}+\frac{x_n}{h}\right)u_{n-1}^k+\left(\frac{t_k^2}{\tau^2}\right)u_n^{k-1}=f_n^k.$$

Then, the last formula can be written as

$$au_n^{k+1}+bu_{n+1}^k+cu_n^k+du_{n-1}^k+eu_n^{k-1}=f_n^k. \quad (9)$$

Here,

$$a=\frac{t_k^2}{\tau^2}+2\frac{t_k}{\tau},\ b=-\frac{x_n^2}{h^2}-\frac{x_k}{h},$$

$$c=-2\frac{t_k^2}{\tau^2}-2\frac{t_k}{\tau}+1+2\frac{x_n^2}{h^2},$$

$$d=-\frac{x_n^2}{h^2}+\frac{x_n}{h} \text{ and } e=\frac{t_k^2}{\tau^2}.$$

From the formula (9), the following matrices' formulas are obtained as

$$AU^{k+1}+BU^k+CU^{k-1}=\phi^k. \quad (10)$$

where, A, B and C are $(N+1)\times(N+1)$ matrix, U^{k+1}, U^k, U^{k-1} and $\phi^k=F_n^k$ is $(N+1)\times 1$ vector as the following

$$A=a\begin{bmatrix} 0&0&0&\dots&0&0&0\\ 0&1&0&\dots&0&0&0\\ 0&0&1&\dots&0&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ 0&0&0&\dots&1&0&0\\ 0&0&0&\dots&0&1&0\\ 0&0&0&\dots&0&0&0 \end{bmatrix}_{(N+1)\times(N+1)}, \quad (11)$$

$$B=\begin{bmatrix} c&b&0&0&\dots&0&0&0&0\\ d&c&b&0&\dots&0&0&0&0\\ 0&d&c&b&\dots&0&0&0&0\\ 0&0&d&c&\dots&0&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\ 0&0&0&0&\dots&c&b&0&0\\ 0&0&0&0&\dots&d&c&b&0\\ 0&0&0&0&\dots&0&d&c&b\\ 0&0&0&0&\dots&0&0&d&c \end{bmatrix}_{(N+1)\times(N+1)} \quad (12)$$

$$C=e\begin{bmatrix} 1&0&0&\dots&0&0&0\\ 0&1&0&\dots&0&0&0\\ 0&0&1&\dots&0&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ 0&0&0&\dots&1&0&0\\ 0&0&0&\dots&0&1&0\\ 0&0&0&\dots&0&0&1 \end{bmatrix}_{(N+1)\times(N+1)}, \quad (13)$$

$$U^{k-1}=\begin{bmatrix} u_0^{k-1}\\ u_1^{k-1}\\ u_2^{k-1}\\ \vdots\\ u_{N-1}^{k-1}\\ u_N^{k-1} \end{bmatrix}_{(N+1)\times 1} \qquad U^k=\begin{bmatrix} u_0^k\\ u_1^k\\ u_2^k\\ \vdots\\ u_{N-1}^k\\ u_N^k \end{bmatrix}_{(N+1)\times 1}$$

$$U^{k+1}=\begin{bmatrix} u_0^{k+1}\\ u_1^{k+1}\\ u_2^{k+1}\\ \vdots\\ u_{N-1}^{k+1}\\ u_N^{k+1} \end{bmatrix}_{(N+1)\times 1.}$$

Modified Gauss elimination method is applied to solve the above difference equations. After that, a solution of the matrix equation is looked for as the following form

$$u_j=\alpha_{j+1}u_{j+1}+\beta_{j+1};\ u_M=0;\ j=M-1,\dots,2,1. \quad (14)$$

Using boundary conditions, the formula

$$u_0=\alpha_1u_1+\beta_1=0$$

is obtained. Then, α_1 is obtained the $(N+1)\times(N+1)$ zero matrix and β_1 is obtained the $(N+1)\times 1$ zero column vector. Using the formula (14), the following formula is found

$$Au_{j+1}+B[\alpha_{j+1}u_{j+1}+\beta_{j+1}]+C[\alpha_ju_j+\beta_j]=\phi_j,$$

,

$$Au_{j+1}+B[\alpha_{j+1}u_{j+1}+\beta_{j+1}]+C[\alpha_j[\alpha_{j+1}u_{j+1} + \beta_{j+1}]+\beta_j]=\phi_j,$$

$$Au_{j+1}+B\alpha_{j+1}u_{j+1}+B\beta_{j+1}+C\alpha_j\alpha_{j+1}u_{j+1} + C\alpha_j\beta_{j+1}+C\beta_j=\phi_j,$$

$$[A+B\alpha_{j+1}+C\alpha_j\alpha_{j+1}]u_{j+1}+B\beta_{j+1} \\ +C\alpha_j\beta_{j+1}+C\beta_j=\phi_j,$$

and then also

$$\begin{aligned} &[A+B\alpha_{j+1}+C\alpha_j\alpha_{j+1}]u_{j+1}=0 \\ &\text{and} \\ &B\beta_{j+1}+C\alpha_j\beta_{j+1}+C\beta_j=\phi_j. \end{aligned} \tag{15}$$

From the (15), the formulas are found

$$\alpha_{j+1}=-(B+C\alpha_j)^{-1}A,$$

and

$$\beta_{j+1}=(B+C\alpha_j)^{-1}(D\phi-C\beta_j),\quad j=1,2,\ldots,M-1.$$

Here, α_j is $(N+1)\times(N+1)$ zero matrix and β_j is $(N+1)\times 1$ zero column vector.

Now, we shall prove the stability estimate by applying the method of analyzing the eigenvalues of the iteration matrices of the schemes for the formula (4). For this, we express $\|A\|=\|A\|_\infty=\max\limits_{1\le k\le N-1}\left[\sum_{i=1}^{N-1}|a_{km}|\right]$, where $A=[a_{km}]_{(N-1)\times(N-1)}$, I is unit matrix.

Let $\rho(A)$ be the spectral radius of a matrix A, which means the maximum of the absolute value of the eigenvalues of the matrix A. We can write the following theorem.

Theorem 1. *If* $-2\frac{t_k^2}{\tau^2}-2\frac{t_k}{\tau}+1+2\frac{x_n^2}{h^2}>0$, *then, the difference scheme (4) is stable.*

Proof. From the method [18], we should prove that $\rho(\alpha_n)<1$, $1\le n\le M$.

$\rho(\alpha_1)=0<1$ is clearly.

$$\rho(\alpha_2)=\left\|-BA^{-1}\right\|\le\|-B\|\left\|A^{-1}\right\| \\ =\|B\|\frac{1}{\min\limits_{1\le k\le N-1}\left\{|a_{kk}|-\sum\limits_{\substack{m\ne k,\\ m=1}}^{N-1}|a_{km}|\right\}}$$

$$=\frac{\left|-2\frac{t_k^2}{\tau^2}-2\frac{t_k}{\tau}+1+2\frac{x_n^2}{h^2}\right|}{\left|\frac{t_k^2}{\tau^2}+\frac{t_k}{\tau}\right|}$$

$$+\frac{\left|-\frac{x_n^2}{h^2}-\frac{x_k}{h}\right|+\left|-\frac{x_n^2}{h^2}+\frac{x_k}{h}\right|}{\left|\frac{t_k^2}{\tau^2}+\frac{t_k}{\tau}\right|}$$

$$=\frac{-2\frac{t_k^2}{\tau^2}-2\frac{t_k}{\tau}+1+2\frac{x_n^2}{h^2}-\frac{x_n^2}{h^2}-\frac{x_k}{h}-\frac{x_n^2}{h^2}+\frac{x_k}{h}}{\frac{t_k^2}{\tau^2}+\frac{t_k}{\tau}}$$

$$=\frac{1-2\frac{t_k^2}{\tau^2}-2\frac{t_k}{\tau}}{\frac{t_k^2}{\tau^2}+\frac{t_k}{\tau}}$$

$$=\frac{1-2(k^2+k)}{(k^2+k)}\le 1,\ k=1,2,...M.$$

If $\rho(\alpha_n)<1$, let us calculate $\rho(\alpha_{n+1})$ for the formula (3) and procedure [19]. We know that $\alpha_{ni}=\rho(\alpha_n)$ and $0\le\rho(\alpha_n)<1$ for $2\le i\le N+1$. Then, we can obtain that $\rho(\alpha_{n+1})<1$. Thus, the proof of the theorem is completed. □

For the stability estimate of the second order difference schemes formula (5), a similar procedure can be used. The stability estimates of the formulas (6) and (7) were given in the [13], [17].

Now let's find the approximate solutions of a few examples for the application of these theoretical expressions.

3. Numerical experiments

In this section, some numerical example for the telegraph partial differential equation by the first and second order difference schemes method will be present. We can calculate the maximum norm of the error of the numerical solution as

$$E_M^N=\max_{1\le k\le N-1,1\le n\le M-1}|u(t_k,x_n)-u_n^k|.$$

Where $u(t_k,x_n)$ represents the exact solution and u_n^k represents numerical solution at points (t_k,x_n). Result of calculations tell us the second order has more accurate than the first order of accuracy difference scheme.

Example 1. *Consider the following initial boundary value problem for Telegraph partial differential equation*

$$\begin{cases} u_{tt}(t,x) + u_t(t,x) - u_{xx}(t,x) - u_x(t,x) + u(t,x) \\ = \cos(x-t) - \sin(x)\cos(t), \ 0 < t < 1, \quad 0 < x < \pi, \\ u(0,x) = -\sin(x), \quad u_t(0,x) = 0, \\ u(t,0) = u(t,\pi) = 0, 0 \le t \le 1, \ 0 \le x \le \pi. \end{cases} \quad (16)$$

Using the Laplace transform method, the exact solution of the problem (16) is $u(x,t) = -\sin(x)\cos(t)$. *Error analysis Table 1 is shown the approximation solution of the problem (16).*

Table 1. Error analysis for exact and approximation solution for example 16.

$\tau = 1/N$, $h = \pi/M$	First Order Difference Scheme	Second Order Difference Scheme
$N = M = 20$	1.1102×10^{-2}	1.8527×10^{-3}
$N = M = 50$	3.8794×10^{-3}	2.9979×10^{-4}
$N = M = 100$	1.8400×10^{-3}	7.5204×10^{-5}
$N = M = 200$	8.9448×10^{-4}	1.8815×10^{-5}
$N = M = 400$	4.4078×10^{-4}	4.7025×10^{-6}
$N = M = 600$	2.9241×10^{-4}	2.0896×10^{-6}

Example 2. *Investigate the following initial boundary value problem for Telegraph partial differential equation*

$$\begin{cases} u_{tt}(t,x) + u_t(t,x) - u_{xx}(t,x) - u_x(t,x) + u(t,x) \\ = (x^2 - 2x - 2)e^{-t} + \pi(1-x)e^{-t}, \\ 0 < t < 1, \ 0 < x < \pi, \\ u(0,x) = x(x-\pi), \quad u_t(0,x) = -x(x-\pi), \\ u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le 1, \ 0 \le x \le \pi. \end{cases} \quad (17)$$

The exact solution of the problem (17) is $u(x,t) = (x^2 - \pi x)e^{-t}$. Error analysis Table 2 is shown the approximation solution of the problem (17).

Table 2. Error analysis for exact and approximation solution for example 17.

$\tau = 1/N$, $h = \pi/M$	First Order Difference Scheme	Second Order Difference Scheme
$N = M = 20$	3.7052×10^{-2}	2.1852×10^{-3}
$N = M = 50$	1.5780×10^{-2}	3.5362×10^{-4}
$N = M = 100$	8.0644×10^{-3}	8.8693×10^{-5}
$N = M = 200$	4.0783×10^{-3}	2.2207×10^{-5}
$N = M = 400$	2.0505×10^{-3}	5.5558×10^{-6}
$N = M = 600$	1.3695×10^{-3}	2.4698×10^{-6}

The exact and approximate solution of these examples are also presented in the following figures.

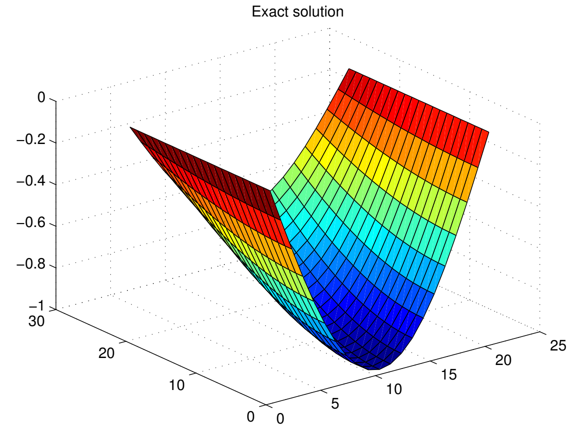

Figure 1. Figure of exact solution for problem16, where N=M=20.

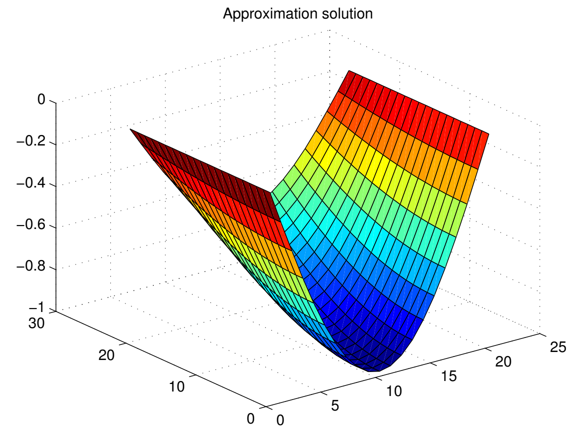

Figure 2. Figure of approximation solution for problem 16, where N=M=20.

Remark 1. *Using the first order difference scheme formula (4), we obtain the the following numerical results for the problem (2) and example (17). For example; Taking* $N = 21, M = 20$, *we obtain* $maxerror = 8.7021 \times 10^{-1}$. *For these values, the figures are the added as follow:*

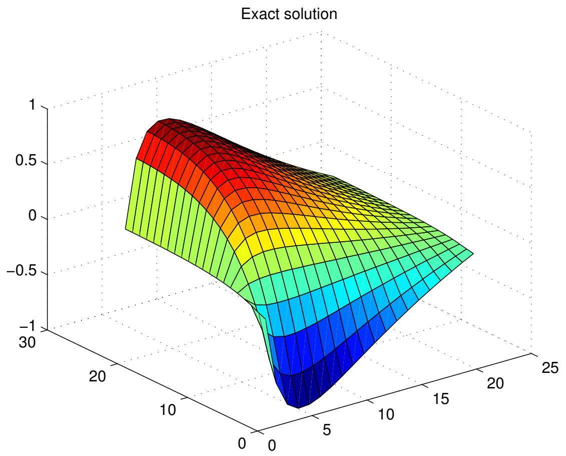

Figure 3. Figure of exact solution for problem(2) and example (16), where N=21, M=20.

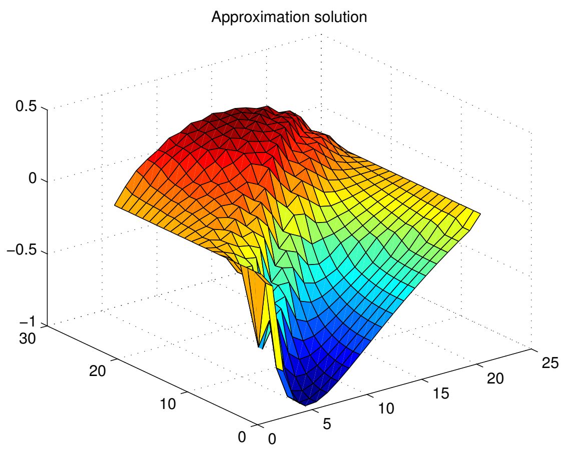

Figure 4. Figure of approximation solution for problem(2) and example (16), where N=21, M=20.

Remark 2. *The following results are obtained through using the Cauchy-Euler formula:*
i. The non-uniform region becomes a smooth region. And this is easier made calculation of the Matlab program.
ii. This also provides to obtain more appropriate and beautiful numerical results.

4. Conclusion

In this paper, the variable telegraph partial differential equation has been investigated. Then, this equation is transformed to the constant coefficient via using Cauchy-Euler formula. For this equation, we construct the first and second order difference schemes. Stability estimate is proved for these difference schemes. The exact and approximate solution of the problem were compared to obtain the error analysis in the maximum norm. Numerical examples show that this method is appropriate for this problem.

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