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RESEARCH ARTICLE

Hermite-Hadamard type inequalities for p-convex stochastic processes

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ABSTRACT

In this study are investigated p-convex stochastic processes which are extensions of convex stochastic processes. A suitable example is also given for this process. In addition, in this case a p-convex stochastic process is increasing or decreasing, the relation with convexity is revealed. The concept of inequality as convexity has an important place in literature, since it provides a broader setting to study the optimization and mathematical programming problems. Therefore, Hermite-Hadamard type inequalities for p-convex stochastic processes and some boundaries for these inequalities are obtained in present study. It is used the concept of mean-square integrability for stochastic processes to obtain the above mentioned results.

1. Introduction and preliminaries

The convexity for stochastic processes is of great importance in optimization, especially in optimal designs, and also useful for numerical approximations when there exist probabilistic quantities in the literature.

In 1980 Nikodem defined convex stochastic processes and gave some properties which are also known for classical convex functions. Some types of convex stochastic processes were introduced by Skowronski in 1992. In 2012 Kotrys obtained the classical Hermite-Hadamard inequality to convex stochastic processes. There are many studies in recent years on the above mentioned processes. A lot of definitions of various convexity and some new inequalities were for these stochastic processes in the literature [7-13].

The author's findings led to our motivation to build our work.The main goal is to introduce p-convex stochastic processes. Moreover, we prove Hermite-Hadamard type inequalities for p-convex stochastic processes and obtain some important results for these processes.

Let us show the definition of a stochhastic process:

Definition 1 ([5]). *The process $\{X(t): t \in I\}$ is a parameterized collection of random variables defined on a common probability space$(\Omega, \Im, P)$. Its parameter t is considered to be time. Then $X(t)$, which can also be shown as $X(t,\omega)$ for $\omega \in \Omega$, is considered to be state or position of the process at time t. For any fixed outcome ω of sample spaceΩ, the deterministic mapping $t \to X(t,\omega)$ denotes a realization, trajectory or sample path of the process. For any particular $t \in I$the mapping depends ω alone, i.e., then we obtain a random variable. It can be said that, $X(t,\omega)$ changes in time in a random manner. We restrict our attention to continuous time stochastic processes, i.e., index set is I: $[0,\infty)$.*

Definition 2 ([5]). *The process $X: I \subset \mathbb{R} \times \Omega \to \mathbb{R}$ is called convex stochastic process if*

$$X(\lambda u + (1-\lambda)v,\cdot) \leq \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot) \text{ (a.e.)}$$

for all $u, v \in I, \lambda \in [0,1]$. If the above inequality is reversed, then $X(t,\cdot)$ is called concave.

Let us give some basic definitions:

Definition 3 ([5]). *The process $X: I \times \Omega \to \mathbb{R}$ is called*
(i) *continuous in probability in I if for all $t_0 \in I$ if*

$$P - \lim_{t \to t_o} X(t,\cdot) = X(t_0,\cdot)$$

where $P - lim$ denotes limit in probability,

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(ii) *mean-square continuous in* I *if for all* $t_0 \in I$ *if*

$$\lim_{t \to t_o} E[X(t,\cdot) - X(t_0,\cdot)]^2 = 0$$

where $E[X(t,\cdot)]$ *denotes expectation value of the random variable* $X(t,\cdot)$*,*

(iii) *increasing (decreasing) if for all* $u, v \in I$ *such that* $u < v$ *if*

$$X(u,\cdot) \leq X(v,\cdot) \quad (X(u,\cdot) \geq X(v,\cdot)) \text{ (a.e.)},$$

(iv) *mean-square differentiable at a point* $t \in I$ *if there is random variable* $X'(t,\cdot): I \times \Omega \to \mathbb{R}$ *such that*

$$X'(t,\cdot) = P - \lim_{t \to t_o} \frac{X(t,\cdot) - X(t_0,\cdot)}{t - t_0}$$

The stochastic process $X: I \times \Omega \to \mathbb{R}$ *is continuous (differentiable) if it is continuous (differentiable) at every point of interval* I*.*

The concept "mean-square convergence" is used as the statement "almost everywhere" in this paper.

Definition 4 ([5]). *Let* $X: I \times \Omega \to \mathbb{R}$ *be Theprocess with* $E[X(t)^2] < \infty$ *and* $u = t_0 < t_1 < \cdots < t_n = v$ *be a partition of* $[u, v] \subset I$*,* $\Theta_k \in [t_{k-1}, t_k]$*,* $k = 1, \ldots, n$*. A random variable* $Y: \Omega \to \mathbb{R}$ *is called mean-square integral of the process* $X(t,\cdot)$ *on* $[u, v]$ *if the following identity holds:*

$$\lim_{n \to \infty} E[\left(\sum_{k=1}^{n} X(\Theta_k) \cdot (t_k - t_{k-1}) - Y\right)^2] = 0.$$

Then we can write $\int_u^v X(t,\cdot)dt = Y(\cdot)$*(a.e..).*

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process X*.*

Now, we give the well-known Hermite-Hadamard integral inequality for convex stochastic processes:

Theorem 1 ([5]). *If* $X: I \times \Omega \to \mathbb{R}$ *is a Jensen-convex stochastic process and mean square continuous in the interval* I*, then we have almost everywhere*

$$X\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{v-u} \int_u^v X(t,\cdot)dt \leq \frac{X(u,\cdot) + X(v,\cdot)}{2}$$

for any $u, v \in I$*,* $u < v$*.*

Definition 5 ([7]). *Let* $I \subset \mathbb{R} \backslash \{0\}$ *be a real interval. The process* $X: I \times \Omega \to \mathbb{R}$ *is called a harmonically convex stochastic process, if the following inequality holds almost everywhere:*

$$X\left(\frac{uv}{\lambda u + (1-\lambda)v},\cdot\right) \leq \lambda X(v,\cdot) + (1-\lambda)X(u,\cdot)$$

for all $u, v \in I$ *and* $\lambda \in [0,1]$*. If the above inequality is reversed, then* X *is called a harmonically concave stochastic process.*

Definition 6 ([13]). *Let* I *be a p-convex set. A function* $f: I \to \mathbb{R}$ *is called a p-convex function or belongs to the class* $PC(I)$*, if the following inequality holds:*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ *and* $t \in [0,1]$*.*

Theorem 2 ([12]). *Let* $f: I \subset (0, \infty) \to \mathbb{R}$ *be a p-convex function. Iff* $\in L[a, b]$*,* $a, b \in I$*,* $a < b$*, then we have*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \leq \frac{f(a) + f(b)}{2}.$$

Remark 1 ([9]). *Let us define the following functions:*

(1) The Beta function:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1} d\lambda.$$

(2) The hypergeometric function $c > b > 0$*;* $|z| < 1$*:*
:

$$ {}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 \lambda^{b-1}(1-\lambda)^{c-b-1}(1-z\lambda)^{-a} d\lambda.$$

2. Main results

The main subject of this paper is to adapt some well-known related results p-convex functions on p-convex stochastic processes. Also, we purpose to obtain Hermite-Hadamard type inequalities for p-convex stochastic processes.

Definition 7. *Let* I *be a p-convex set. The process* $X: I \times \Omega \to \mathbb{R}$ *is called a p-convex stochastic process, if the following inequality holds almost everywhere:*

$$X\left([\lambda u^p + (1-\lambda)v^p]^{\frac{1}{p}},\cdot\right) \leq \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot)$$

for all $u, v \in I$ *and* $\lambda \in [0,1]$*.*

Remark 2. *The interval* I *is called a p-convex set, if* $[\lambda u^p + (1-\lambda)v^p]^{\frac{1}{p}} \in I$ *for all* $u, v \in I$ *and* $\lambda \in [0,1]$*, where* $p = 2k+1$ *or,* $p = \frac{n}{m} n = 2r+1$*,* $m = 2t+1$ *and* $k, r, t \in \mathbb{N}$*.*

Remark 3. *If* $I \subset (0, \infty)$ *and* $p \in \mathbb{R} \backslash \{0\}$*, then* $[\lambda u^p + (1-\lambda)v^p]^{\frac{1}{p}} \in I$ *for all* $u, v \in I$ *and* $\lambda \in [0,1]$*.*

Thus, we can also define p-convex stochastic processes as follows:

Definition 8. *The process* $X: I \times \Omega \to \mathbb{R}$ *is called a p-convex stochastic process, if the following inequality holds almost everywhere:*

$$X\left([\lambda u^p + (1-\lambda)v^p]^{\frac{1}{p}},\cdot\right) \leq \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot) \quad (1)$$

for all $u, v \in I \subset (0, \infty)$*,* $\lambda \in [0,1]$*,* $p \in \mathbb{R} \backslash \{0\}$*. If the inequality in Eq. (1) is reversed, then the process* X *is called p-concave.*

According to Definition 8, it can be easily seen that for $p = 1$ or $p = -1$, a p-convex stochastic process reduces to convex and harmonically convex stochastic process on $I \subset (0, \infty)$, respectively.

Example 1. *Let* $X: (0, \infty) \times \Omega \to \mathbb{R}$*,* $X(u,\cdot) = u^p$*,* $p \neq$

0 and $Y:(0,\infty)\times\Omega\to\mathbb{R}$, $Y(u,\cdot)=c$, $c\in\mathbb{R}$, then X and Y are both p-convex and p-concave stochastic processes.

Lemma 1. *Let $X:I\subset(0,\infty)\times\Omega\to\mathbb{R}$ be a p-convex stochastic process and mean-square integrable on I°. Then the following equality holds almost everywhere:*

$$\frac{p}{v^p-u^p}\int_u^v\frac{X(t,\cdot)}{t^{1-p}}dt$$

$$=\frac{1}{v-u}\int_u^v X\left(\left[\frac{y-u}{v-u}v^p+\frac{v-y}{v-u}u^p\right]^{\frac{1}{p}},\cdot\right)dy$$

for all $u,v\in I$, $p\in\mathbb{R}\backslash\{0\}$.

Proof. Changing of $t^p=\frac{y-u}{v-u}v^p+\frac{v-y}{v-u}u^p$ in

$$\frac{p}{v^p-u^p}\int_u^v\frac{X(t,\cdot)}{t^{1-p}}dt,$$

then the proof of Lemma 1 is completed.

Theorem 3. *Let $X:I\subset(0,\infty)\times\Omega\to\mathbb{R}$ be a p-convex stochastic process. If X is mean-square integrable on $[u,v]$, then we have almost everywhere*

$$X\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}},\cdot\right)\le\frac{p}{v^p-u^p}\int_u^v\frac{X(t,\cdot)}{t^{1-p}}dt\le\frac{X(u,\cdot)+X(v,\cdot)}{2}$$

for all $u,v\in I$, $u<v$.

Proof. Changing of $\lambda=\frac{v-y}{v-u}$ in Eq. (1), we get

$$X\left(\left[\frac{v-y}{v-u}u^p+\frac{y-u}{v-u}v^p\right]^{\frac{1}{p}},\cdot\right)$$

$$\le\left(\frac{v-y}{v-u}\right)X(u,\cdot)+\left(\frac{y-u}{v-u}\right)X(v,\cdot).$$

Integrating on $[u,v]$ and using Lemma 1, we have

$$\frac{1}{v-u}\int_u^v X\left(\left[\frac{y-u}{v-u}v^p+\frac{v-y}{v-u}u^p\right]^{\frac{1}{p}},\cdot\right)dy$$

$$=\frac{p}{v^p-u^p}\int_u^v\frac{X(t,\cdot)}{t^{1-p}}dt\le\frac{X(u,\cdot)+X(v,\cdot)}{2}.$$

Changing of $y=\frac{1}{2}(u+v)+t$ in Lemma 1, we obtain

$$\frac{p}{v^p-u^p}\int_u^v\frac{X(t,\cdot)}{t^{1-p}}dt$$

$$=\frac{1}{v-u}\int_{-\frac{1}{2}(v-u)}^{\frac{1}{2}(v-u)}X\left(\left[\begin{array}{l}\frac{1}{2}(u^p+v^p)\\+\frac{v^p-u^p}{v-u}t\end{array}\right]^{\frac{1}{p}},\cdot\right)dt$$

$$\ge\frac{2}{v-u}\int_0^{\frac{1}{2}(v-u)}X\left(\left[\frac{1}{2}(u^p+v^p)\right]^{\frac{1}{p}},\cdot\right)dt$$

$$=X\left(\left[\frac{1}{2}(u^p+v^p)\right]^{\frac{1}{p}},\cdot\right)\text{ (a.e.).}$$

Corollary 1. *If X is mean-square integrable on $[u,v]$, then we have almost everywhere*

$$X\left(\frac{2uv}{u+v},\cdot\right)\le\frac{uv}{v-u}\int_u^v\frac{X(t,\cdot)}{t^2}dt\le\frac{X(u,\cdot)+X(v,\cdot)}{2}.$$

Proof. In Theorem 3, if $p=-1$, then the proof of Corollary 1 is completed.

Lemma 2. *Let $X:I\subset(0,\infty)\times\Omega\to\mathbb{R}$ be a mean-square differentiable stochastic process on I°. If X' is mean-square integrable on $[u,v]$, then we have almost everywhere*

$$\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{p}{v^p-u^p}\int_u^v\frac{X(t,\cdot)}{t^{1-p}}dt$$

$$=\frac{v^p-u^p}{2p}\int_0^1\left[\begin{array}{c}\frac{1-2\lambda}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}\\ \times X'\left([\lambda u^p+(1-\lambda)v^p]^{\frac{1}{p}},\cdot\right)\end{array}\right]d\lambda$$

for all $u,v\in I$, $p\in\mathbb{R}\backslash\{0\}$.

Proof. It suffices to show that

$$\int_0^1\left[\begin{array}{c}\frac{1-2\lambda}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}\\ \times X'\left([\lambda u^p+(1-\lambda)v^p]^{\frac{1}{p}},\cdot\right)\end{array}\right]d\lambda$$

$$=\frac{X(u,\cdot)+X(v,\cdot)}{v^p-u^p}-\frac{2p}{v^p-u^p}\int_0^1 X\left(\left[\begin{array}{c}\lambda u^p\\+(1-\lambda)v^p\end{array}\right]^{\frac{1}{p}},\cdot\right)d\lambda$$

$$=\frac{X(u,\cdot)+X(v,\cdot)}{v^p-u^p}-\frac{2p^2}{(v^p-u^p)^2}\int_u^v\frac{X(t,\cdot)}{t^{1-p}}dt.$$

Multiplying by $\frac{v^p-u^p}{2p}$ both sides of above equality then the proof of Lemma 2 is completed.

Corollary 2. *If X' is mean-square integrable on $[u,v]$, then the following equality holds almost everywhere:*

$$\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{1}{v-u}\int_u^v X(t,\cdot)dt$$

$$=\frac{v-u}{2}\int_0^1(1-2\lambda)X'\big((\lambda u+(1-\lambda)v),\cdot\big)d\lambda.$$

Proof. In Lemma 2, if we take $p=1$, then the proof of Corollary 2 is completed.

Corollary 3. *If X' is mean-square integrable on $[u,v]$, then almost everywhere*

$$\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{uv}{v-u}\int_u^v\frac{X(t,\cdot)}{t^2}dt$$

$$=\frac{uv(v-u)}{2}\int_0^1\frac{1-2\lambda}{[\lambda v+(1-\lambda)u]^2}X'\left(\frac{uv}{\lambda v+(1-\lambda)u},\cdot\right)d\lambda$$

Proof. In Lemma 2, if we take $p=-1$, then the proof of Corollary 3 is completed.

Theorem 4. *Let $X:I\subset(0,\infty)\times\Omega\to\mathbb{R}$ be a differentiable stochastic process on I° and X' be mean-square integrable on $[u,v]$. If $|X'|^q$ is a p-convex stochastic process on $[u,v]$ for $q\ge1$, then the following inequality holds almost everywhere:*

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{p}{v^p-u^p}\int_u^v\frac{X(t,\cdot)}{t^{1-p}}dt\right|$$

$$\le\frac{v^p-u^p}{2p}C_1^{1-\frac{1}{q}}\left[C_2\left|X'(u,\cdot)\right|^q+C_3\left|X'(v,\cdot)\right|^q\right]^{\frac{1}{q}}$$

for all $u, v \in I^\circ$, $u < v$, $p \in \mathbb{R}\backslash\{0\}$ *and where*

$$C_1 = C_1(u,v;p) = \frac{1}{4}\left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}-1} \times \begin{bmatrix} {}_2F_1\left(1-\frac{1}{p},2;3;\frac{u^p-v^p}{u^p+v^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},2;3;\frac{v^p-u^p}{u^p+v^p}\right)\end{bmatrix},$$

$$C_2 = C_2(u,v;p) = \frac{1}{24}\left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}-1} \times \begin{bmatrix} {}_2F_1\left(1-\frac{1}{p},2;4;\frac{u^p-v^p}{u^p+v^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},2;4;\frac{v^p-u^p}{u^p+v^p}\right)\end{bmatrix},$$

$$C_3 = C_3(u,v;p) = C_1 - C_2.$$

Proof. Using the power mean integral inequality and Lemma 2, then we have

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2} - \frac{p}{v^p-u^p}\int_u^v \frac{X(t,\cdot)}{t^{1-p}}dt\right|$$

$$\le \frac{v^p-u^p}{2p}\int_0^1 \left[\left|\frac{1-2\lambda}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}\right| \times \left|X'\left([\lambda u^p+(1-\lambda)v^p]^{\frac{1}{p}},\cdot\right)\right|\right]d\lambda$$

$$\le \frac{v^p-u^p}{2p}\left(\int_0^1 \frac{|1-2\lambda|}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}d\lambda\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left[\frac{|1-2\lambda|}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}} \times \left|X'\left([\lambda u^p+(1-\lambda)v^p]^{\frac{1}{p}},\cdot\right)\right|^q\right]d\lambda\right)^{\frac{1}{q}}.$$

Hence, using p-convexity of the stochastic process $|X'|^q$on $[u,v]$, we have almost everywhere

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2} - \frac{p}{v^p-u^p}\int_u^v \frac{X(t,\cdot)}{t^{1-p}}dt\right|$$

$$\le \frac{v^p-u^p}{2p}\left(\int_0^1 \frac{|1-2\lambda|}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}d\lambda\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{|1-2\lambda||\lambda X'(u,\cdot)+(1-\lambda)X'(v,\cdot)|^q}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}d\lambda\right)^{\frac{1}{q}}$$

$$\le \frac{v^p-u^p}{2p}C_1^{1-\frac{1}{q}}\left[C_2|X'(u,\cdot)|^q + C_3|X'(v,\cdot)|^q\right]^{\frac{1}{q}},$$

where $\int_0^1 \frac{|1-2\lambda|}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}d\lambda = C_1(u,v;p)$,

$$\int_0^1 \frac{|1-2\lambda|\lambda}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}d\lambda = C_2(u,v;p),$$

$$\int_0^1 \frac{|1-2\lambda|(1-\lambda)}{[\lambda u^p+(1-\lambda)v^p]^{1-\frac{1}{p}}}d\lambda = C_1(u,v;p) - C_2(u,v;p).$$

Corollary 4. *If* $|X'|^q$ *is a p-convex stochastic process on* $[u,v]$, *then almost everywhere*

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2} - \frac{p}{v^p-u^p}\int_u^v \frac{X(t,\cdot)}{t^{1-p}}dt\right| \le \frac{v^p-u^p}{2p}\left[C_2|X'(u,\cdot)| + C_3|X'(v,\cdot)|\right],$$

where C_2 *and* C_3 *are defined as in Theorem 4.*

Proof. If $q = 1$ in Theorem 4, then the proof of Corollary 4 is completed.

Theorem 5. *Let* $X: I^\circ \subset \mathbb{R}\times\Omega \to \mathbb{R}$ *be a differentiable stochastic process on* I. *If* $|X'|$ *is a convex stochastic process on* $[u,v]$, *then the following inequality holds almost everywhere:*

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2} - \frac{1}{v-u}\int_u^v X(t,\cdot)dt\right| \le \frac{(v-u)(|X'(u,\cdot)|+|X'(v,\cdot)|)}{8}$$

for all $u, v \in I^\circ$, $u < v$.

Proof. If $p = 1$ in Corollary 4, then then the proof of Theorem 5 is completed.

Corollary 5. *If* $|X'|^q$ *is harmonically convex stochastic process on* $[u,v]$ *for* $q \ge 1$, *then almost everywhere*

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2} - \frac{uv}{v-u}\int_u^v \frac{X(t,\cdot)}{t^2}dt\right| \le \frac{uv(v-u)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2|X'(u,\cdot)|^q + \lambda_3|X'(v,\cdot)|^q\right]^{\frac{1}{q}},$$

where $\lambda_1 = \frac{1}{uv} - \frac{2}{(v-u)^2}ln\left(\frac{(u+v)^2}{4uv}\right)$,

$$\lambda_2 = \frac{-1}{v(v-u)} + \frac{3u+v}{(v-u)^3}ln\left(\frac{(u+v)^2}{4uv}\right),$$

$$\lambda_2 = \frac{1}{u(v-u)} - \frac{3v+u}{(v-u)^3}ln\left(\frac{(u+v)^2}{4uv}\right) = \lambda_1 - \lambda_2.$$

Proof. If $p = -1$ in Theorem 4, then the proof of Corollary 5 is completed.

Theorem 6. *Let* $X: I \subset (0,\infty)\times\Omega \to \mathbb{R}$ *be a differentiable stochastic process on* I° *and mean-square integrable on* $[u,v]$. *If* $|X'|^q$ *is a p-convex stochastic process on* $[u,v]$ *for* $q > 1, \frac{1}{r}+\frac{1}{q} = 1$, *then almost everywhere*

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2} - \frac{p}{v^p-u^p}\int_u^v \frac{X(t,\cdot)}{t^{1-p}}dt\right| \le \frac{v^p-u^p}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[C_4|X'(u,\cdot)|^q + C_5|X'(v,\cdot)|^q\right]^{\frac{1}{q}} \quad (2)$$

for all $u, v \in I^\circ$, $u < v$, $p \in \mathbb{R}\backslash\{0\}$ *and where*

$$C_4 = C_4(u,v;p;q) = \begin{cases} \frac{1}{2u^{qp-q}}\ {}_2F_1\left(q-\frac{q}{p},1;3;1-\left(\frac{v}{u}\right)^p\right), & p < 0, \\ \frac{1}{2v^{qp-q}}\ {}_2F_1\left(q-\frac{q}{p},2;3;1-\left(\frac{u}{v}\right)^p\right), & p > 0\end{cases},$$

$$C_5 = C_5(u,v;p;q) =$$

$$\begin{cases} \frac{1}{2u^{qp-q}} {}_2F_1\left(q-\frac{q}{p},2;3;1-\left(\frac{v}{u}\right)^p\right), & p<0, \\ \frac{1}{2v^{qp-q}} {}_2F_1\left(q-\frac{q}{p},1;3;1-\left(\frac{u}{v}\right)^p\right), & p>0 \end{cases}.$$

Proof. Hölder's inequality, using p-convexity of the stochastic process $|X'|^q$on $[u,v]$ from Lemma 2

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{p}{v^p-u^p}\int_u^v \frac{X(t,\cdot)}{t^{1-p}}dt\right|$$

$$\leq \frac{v^p-u^p}{2p}\left(\int_0^1 |1-2\lambda|^r d\lambda\right)^{\frac{1}{r}}$$

$$\times\left(\int_0^1\left[\frac{\left|X'\left([\lambda u^p+(1-\lambda)v^p]^{\frac{1}{p}},\cdot\right)\right|^q}{[\lambda u^p+(1-\lambda)v^p]^{q-\frac{q}{p}}}\right]d\lambda\right)^{\frac{1}{q}}$$

$$\leq \frac{v^p-u^p}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}$$

$$\times\left(\int_0^1 \frac{\lambda|X'(u,\cdot)|^q+(1-\lambda)|X'(v,\cdot)|^q}{[\lambda u^p+(1-\lambda)v^p]^{q-\frac{q}{p}}}d\lambda\right)^{\frac{1}{q}},$$

where $\int_0^1 \frac{\lambda}{[\lambda u^p+(1-\lambda)v^p]^{q-\frac{q}{p}}}d\lambda$

$$=\begin{cases} \frac{1}{2u^{qp-q}} {}_2F_1\left(q-\frac{q}{p},1;3;1-\left(\frac{v}{u}\right)^p\right), & p<0, \\ \frac{1}{2v^{qp-q}} {}_2F_1\left(q-\frac{q}{p},2;3;1-\left(\frac{u}{v}\right)^p\right), & p>0 \end{cases}, \quad (3)$$

$$\int_0^1 \frac{1-\lambda}{[\lambda u^p+(1-\lambda)v^p]^{q-\frac{q}{p}}}d\lambda$$

$$=\begin{cases} \frac{1}{2u^{qp-q}} {}_2F_1\left(q-\frac{q}{p},2;3;1-\left(\frac{v}{u}\right)^p\right), & p<0, \\ \frac{1}{2v^{qp-q}} {}_2F_1\left(q-\frac{q}{p},1;3;1-\left(\frac{u}{v}\right)^p\right), & p>0 \end{cases} \quad (4).$$

Substituting Eq. (3) and (4) in Eq. (2), then the proof is completed.

Corollary 6. *If $|X'|^q$ is a convex stohastic process on $[u,v]$ then the following inequality holds:*

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{1}{v-u}\int_u^v X(t,\cdot)dt\right|$$

$$\leq \frac{(v-u)}{2(p+1)^{\frac{1}{p}}}\left[\frac{|X'(u,\cdot)|^q+|X'(v,\cdot)|^q}{2}\right]^{\frac{1}{q}}, \text{(a.e.)}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. In Theorem 6, if we take $p=1$, then the proof of Corollary 6 is completed.

Corollary 7. *If $|X'|^q$ is a harmonically convex stochastic process on $[u,v]$ for $q>1,\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{uv}{v-u}\int_u^v \frac{X(t,\cdot)}{t^2}dt\right|$$

$$\leq \frac{uv(v-u)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\mu_1|X'(u,\cdot)|^q+\mu_2|X'(v,\cdot)|^q\right]^{\frac{1}{q}},$$

where $\mu_1=\frac{u^{2-2q}+v^{1-2q}[(v-u)(1-2q)-u]}{2(v-u)^2(1-q)(1-2q)}$,

$$\mu_2=\frac{v^{2-2q}-u^{1-2q}[(v-u)(1-2q)+v]}{2(v-u)^2(1-q)(1-2q)}.$$

Proof. In Theorem 6, if we take $p=-1$, then the proof of Corollary 7 is completed.

Theorem 7. *Let $X: I\subset(0,\infty)\times\Omega\to\mathbb{R}$ be a differentiable stochastic process on $I°$ and X'be mean-square integrable on $[u,v]$. If $|X'|^q$ is a p-convex stochastic process on $[u,v]$ for $q>1,\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{p}{v^p-u^p}\int_u^v \frac{X(t,\cdot)}{t^{1-p}}dt\right|$$

$$\leq \frac{v^p-u^p}{2p}C_6^{\frac{1}{r}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\frac{|X'(u,\cdot)|^q+|X'(v,\cdot)|^q}{2}\right]^{\frac{1}{q}} \quad (5)$$

for all $u,v\in I°$, $u<v$,$p\in\mathbb{R}\backslash\{0\}$and where

$$C_6=C_6(u,v;p;r)$$

$$=\begin{cases} \frac{1}{u^{rp-r}} {}_2F_1\left(r-\frac{r}{p},1;2;1-\left(\frac{v}{u}\right)^p\right), & p<0, \\ \frac{1}{v^{rp-r}} {}_2F_1\left(r-\frac{r}{p},1;2;1-\left(\frac{u}{v}\right)^p\right), & p>0 \end{cases}.$$

Proof. Using Hölder's inequality, p-convexity of the stochastic process $|X'|^q$on $[u,v]$ and Lemma 2

$$\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{p}{v^p-u^p}\int_u^v \frac{X(t,\cdot)}{t^{1-p}}dt\right|$$

$$\leq \frac{v^p-u^p}{2p}\left(\int_0^1 \frac{1}{[\lambda u^p+(1-\lambda)v^p]^{r-\frac{r}{p}}}d\lambda\right)^{\frac{1}{r}}$$

$$\times\left(\int_0^1 |1-2\lambda|^q\left|X'\left([\lambda u^p+(1-\lambda)v^p]^{\frac{1}{p}},\cdot\right)\right|^q d\lambda\right)^{\frac{1}{q}}$$

$$\leq \frac{v^p-u^p}{2p}C_6^{\frac{1}{r}}(u,v;p;r)\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\times\left(\frac{|X'(u,\cdot)|^q+|X'(v,\cdot)|^q}{2}\right)^{\frac{1}{q}},$$

where $C_6(u,v;p;r)=\int_0^1 \frac{1}{[\lambda u^p+(1-\lambda)v^p]^{r-\frac{r}{p}}}d\lambda$

$$=\begin{cases} \frac{1}{u^{rp-r}} {}_2F_1\left(r-\frac{r}{p},1;2;1-\left(\frac{v}{u}\right)^p\right), & p<0, \\ \frac{1}{v^{rp-r}} {}_2F_1\left(r-\frac{r}{p},1;2;1-\left(\frac{u}{v}\right)^p\right), & p>0 \end{cases}, \quad (6)$$

$$\int_0^1 |1-2\lambda|^q\lambda d\lambda=\int_0^1 |1-2\lambda|^q(1-\lambda)d\lambda=\frac{1}{2(q+1)}. \quad (7)$$

Substituting Eq. (6) and (7) in Eq. (5), then the proof of Theorem 7 is completed.

3. Conclusion

In this paper, we considered an important extension of convexity for stochastic processes which is called p-convex stochastic processes and obtained new Hermite-Hadamard inequalities for these processes. In the future, new inequalities for the other convex stochastic processes can be obtained using similar methods in this study.

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