IJOCTA
An International Journal of Optimization and Control: Theories & Applications
2010

RESEARCH ARTICLE

Reproducing kernel Hilbert space method for solutions of a coefficient inverse problem for the kinetic equation

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ARTICLE INFO	ABSTRACT
	On the basis of a reproducing kernel Hilbert space, reproducing kernel functions for solving the coefficient inverse problem for the kinetic equation are given in this paper. Reproducing kernel functions found in the reproducing kernel Hilbert space imply that they can be considered for solving such inverse problems. We obtain approximate solutions by reproducing kernel functions. We show our results by a table. We prove the efficiency of the reproducing kernel Hilbert space method for solutions of a coefficient inverse problem for the kinetic equation.

1. Introduction

Kinetic theory emerged with Maxwell and Boltzmann, Hilbert, Enskog, Chapman, Vlasov, and Grad. Investigating for a form of matter which could clarify Saturn's rings, Maxwell considered that they were performed of rocks colliding and gravitating around the planet. The density of matter is then parameterized by the space position x and the velocity v of the rocks. Boltzmann modeled the operation, endowed a common representation of a dilute gas as particles undergoing collisions and with free motion between collisions, and he found the famous equation which is now named after him [1]. Vlasov obtained another kinetic equation (KE) for plasmas of charged particles. Kinetic equations (KEs) rise in a variety of sciences and implementations such as astrophysics, aerospace engineering, nuclear engineering, particle fluid interactions and semi-conductor technology recently. The general property of these models is that the underlying Partial Differential Equation is posed in the phase space $(x, v) \in R^n$ $n \geq 1$, [2].

We consider the problem of obtaining (f, σ) in Ω from the following equation [1]:

$$M_v(x,v)f_x(x,v) \quad - \quad M_x(x,v)f_v(x,v) \quad (1)$$
$$- \quad \sigma(x,v)f = 0.$$

with the boundary conditions:

$$f(a,v) \quad = \quad g(v), \quad f(b,v) = h(v) \quad (2)$$
$$f(x,c) \quad = \quad m(x), \quad f(x,d) = n(x). \quad (3)$$

In this work, the reproducing kernel functions for solving a coefficient inverse problem (IP) for the KE are given. Reproducing kernels were used for the first time at the beginning of the twentieth century by Zaremba in his work on boundary value problems for harmonic and biharmonic functions [3, 4]. The general theory of reproducing kernel Hilbert spaces was established simultaneously and independently by Aronszajn [5] and Bergman [6] in 1950. Mokhtari et al. have investigated an inverse problem for a parabolic equation with a nonlocal boundary condition in the reproducing kernel space [7]. Cui et al. have used

reproducing kernel method of solving the coefficient inverse problem [8]. Xu et al. used simplified reproducing kernel method for fractional differential equations with delay [9]. Tang et al. applied fitted reproducing kernel method for singularly perturbed delay initial value problems [10]. Fardi et al. implemented the reproducing kernel method for some variational problems depending on indefinite integrals [11]. Wahba used regression design for some equivalence classes of kernels [12]. Nashed et al. found regularization and approximation of linear operator equations in reproducing kernel spaces [13]. Al e'damat applied analytical-numerical method for solving a class of two-point boundary value problems [14]. For more details see [15-21].

2. Reproducing kernel functions

In this section, we give some important reproducing kernel functions.

Definition 1. *Hilbert function space H is a reproducing kernel space if and only if for any fixed $x \in X$, the linear functional $I(f) = f(x)$ is bounded [22].*

Definition 2. *We describe the space $T_2^2[1,2]$ as:*

$$T_2^2[1,2] = \{f \in AC[1,2] : f' \in AC[1,2],$$
$$f'' \in L^2[1,2], \ f(1) = 0 = f(2)\}.$$

The inner product and the norm in $T_2^1[1,2]$ are obtained as follow:

$$\langle f,g \rangle_{T_2^2} = \sum_{i=0}^{1} f^{(i)}(1)g^{(i)}(1)$$
$$+ \int_1^2 f''(s)g''(s)\mathrm{d}s,$$
$$f,g \in T_2^2[1,2]$$

and

$$\|f\|_{T_2^2} = \sqrt{\langle f,f \rangle_{T_2^2}}, \quad f \in T_2^2[1,2].$$

Theorem 1. *Reproducing kernel function A_k of reproducing kernel space $T_2^2[1,2]$ is found as follow:*

$$A_k(s) = \begin{cases} \sum_{i=0}^{3} c_i(k)s^i, & s \le k, \\ \\ \sum_{i=0}^{3} d_i(k)s^i, & s > k, \end{cases} \quad (4)$$

where

$$c_0(k) = \frac{7}{12} - \frac{5}{8}k - \frac{1}{24}k^3 + \frac{1}{4}k^2,$$
$$c_1(k) = -\frac{5}{8} - \frac{3}{16}k - \frac{1}{16}k^3 + \frac{3}{8}k^2,$$
$$c_2(k) = \frac{1}{4} + \frac{7}{8}k + \frac{1}{8}k^3 - \frac{3}{4}k^2,$$
$$c_3(k) = -\frac{5}{24} - \frac{1}{16}k - \frac{1}{48}k^3 + \frac{1}{8}k^2,$$
$$d_0(k) = \frac{7}{12} - \frac{5}{8}k - \frac{5}{24}k^3 + \frac{1}{4}k^2,$$
$$d_1(k) = -\frac{5}{8} - \frac{3}{16}k - \frac{1}{16}k^3 + \frac{7}{8}k^2,$$
$$d_2(k) = \frac{1}{4} + \frac{3}{8}k + \frac{1}{8}k^3 - \frac{3}{4}k^2,$$
$$d_3(k) = -\frac{1}{24} - \frac{1}{16}k - \frac{1}{48}k^3 + \frac{1}{8}k^2.$$

Proof. Let $f \in T_2^2[1,2]$ and $1 \le k \le 2$. We have

$$\langle f,A_k \rangle_{T_2^2} = \sum_{i=0}^{1} f^{(i)}(1)A_k^{(i)}(1) + \int_1^2 f''(x)A_k''(x)\mathrm{d}x$$
$$= f(1)A_k(1) + f'(1)A_k'(1)$$
$$+ f'(2)A_k''(2) - f'(1)A_k''(1)$$
$$- f(2)A_k^{(3)}(2) + f(1)A_k^{(3)}(1)$$
$$+ \int_1^2 f(x)A_k^{(4)}(x)\mathrm{d}x.$$

by integration by parts. Then, we get

$$\langle f,A_k \rangle_{T_2^2} = f(k).$$

This completes the proof. $\square$

Definition 3. *We describe the space $M_2^2[-1,1]$ by:*

$$M_2^2[-1,1] = \{g \in AC[-1,1] : g' \in AC[-1,1],$$
$$g'' \in L^2[-1,1], \ g(-1) = 0 = g(1)\}.$$

The inner product and the norm in $M_2^1[-1,1]$ are found as:

$$\langle g,h \rangle_{M_2^2} = \sum_{i=0}^{1} g^{(i)}(-1)h^{(i)}(-1)$$
$$+ \int_1^2 g''(z)h''(z)\mathrm{d}z,$$
$$g,h \in M_2^2[-1,1]$$

and

$$\|g\|_{M_2^2} = \sqrt{\langle g,g \rangle_{M_2^2}},$$
$$g \in M_2^2[-1,1].$$

Theorem 2. *Reproducing kernel function B_p of reproducing kernel space $M_2^2[-1,1]$ is acquired as:*

$$B_p(z) = \begin{cases} \sum_{i=0}^{3} c_i(p)z^i, & z \le p, \\ \\ \sum_{i=0}^{3} d_i(p)z^i, & z > p, \end{cases} \tag{5}$$

where

$$c_0(p) = \frac{31}{240} + \frac{1}{80}p - \frac{17}{80}p^2 + \frac{17}{240}p^3,$$

$$c_1(p) = \frac{1}{80} + \frac{13}{80}p - \frac{21}{80}p^2 + \frac{7}{80}p^3,$$

$$c_2(p) = -\frac{17}{80} + \frac{19}{80}p - \frac{3}{80}p^2 + \frac{1}{80}p^3,$$

$$c_3(p) = -\frac{23}{240} + \frac{7}{80}p + \frac{1}{80}p^3 - \frac{1}{20}p^3,$$

$$d_0(p) = \frac{31}{240} + \frac{1}{80}p - \frac{17}{80}p^2 - \frac{23}{240}p^3,$$

$$d_1(p) = \frac{1}{80} + \frac{13}{80}p + \frac{19}{80}p^2 + \frac{7}{80}p^3,$$

$$d_2(p) = -\frac{17}{800} - \frac{21}{80}p - \frac{3}{80}p^2 + \frac{1}{80}p^3,$$

$$d_3(p) = \frac{17}{240} + \frac{7}{80}p + \frac{1}{80}p^2 - \frac{1}{240}p^3.$$

Proof. Let $g \in M_2^2[-1,1]$ and $-1 \le p \le 1$. We obtain

$$\begin{aligned} \langle g, B_p \rangle_{M_2^2} &= \sum_{i=0}^{1} g^{(i)}(-1)B_p^{(i)}(-1) \\ &+ \int_{-1}^{1} g''(x)B_p''(x)\mathrm{d}x \\ &= g(-1)B_p(-1) + g'(-1)B_p'(-1) \\ &+ g'(1)B_p''(1) - g'(-1)B_p''(-1) \\ &- g(1)B_p^{(3)}(1) + g(-1)B_p^{(3)}(-1) \\ &+ \int_{-1}^{1} g(x)B_p^{(4)}(x)\mathrm{d}x. \end{aligned}$$

by integration by parts. Then, we get

$$\langle g, B_p \rangle_{M_2^2} = g(p).$$

This completes the proof. $\square$

Definition 4. *We describe the space $M_2^1[-1,1]$ as:*

$$M_2^1[-1,1] = \{h \in AC[-1,1] : h' \in L^2[-1,1]\}.$$

The inner product and the norm in $M_2^1[-1,1]$ are given as:

$$\begin{aligned} \langle h, p \rangle_{V_2^1} &= h(-1)p(-1) \\ &+ \int_{-1}^{1} h'(t)p'(t)\mathrm{d}t, \\ h, p &\in M_2^1[-1,1] \end{aligned}$$

and

$$\|h\|_{T_2^1} = \sqrt{\langle h, h \rangle_{T_2^1}},$$
$$h \in M_2^1[-1,1].$$

Lemma 1. *The space $M_2^1[-1,1]$ is a reproducing kernel space, and its reproducing kernel function E_k is given as [15]:*

$$E_k(s) = 2 + s, \quad s \le k,$$
$$2 + k, \quad s > k.$$

Definition 5. *We define the space $T_2^1[1,2]$ by*

$$T_2^1[1,2] = \{h \in AC[1,2] : h' \in L^2[1,2]\}.$$

The inner product and the norm in $T_2^1[1,2]$ are given as:

$$\langle h, p \rangle_{V_2^1} = h(1)p(1) + \int_{1}^{2} h'(t)p'(t)\mathrm{d}t, \quad h, p \in T_2^1[1,2]$$

and

$$\|h\|_{T_2^1} = \sqrt{\langle h, h \rangle_{T_2^1}}, \quad h \in T_2^1[1,2].$$

Lemma 2. *The space $T_2^1[1,2]$ is a reproducing kernel space, and its reproducing kernel function F_p is given as [15]:*

$$F_p(z) = z, \quad z \le p,$$
$$p, \quad z > p.$$

3. Main results

Definition 6. *If $m + n > 2$, define the binary space [22]*

$$W_2^{(m,n)}(\Omega) = \{u : \Omega \to R \,|$$

$$Lu \in W_2^{(1,1)}(\Omega) \text{ if } signature(L) \preceq (m-1, n-1)\}.$$

Equip $W_2^{(m,n)}(\Omega)$ with the inner product

$$\begin{aligned} \langle u, v \rangle_{W_2^{(m,n)}(\Omega)} &= \sum_{i=0}^{m-1} \int_{c}^{d} \frac{\partial^n}{\partial t^n}\frac{\partial^i u}{\partial x^i}(a,t)\frac{\partial^n}{\partial t^n}\frac{\partial^i v}{\partial x^i}(a,t)dt \\ &+ \sum_{j=0}^{n-1} \left\langle \frac{\partial^j u}{\partial t^j}(\cdot,c), \frac{\partial^j v}{\partial t^j}(\cdot,c) \right\rangle_{W_2^m[a,b]} \end{aligned}$$

$$+ \int\int_{\Omega} \frac{\partial^2}{\partial x\partial t}\left(\frac{\partial^{m+n-2}u}{\partial x^{m-1}\partial t^{n-1}}\right)\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial^{m+n-2}v}{\partial x^{m-1}\partial t^{n-1}}\right)dxdt.$$

We found the main reproducing kernel function for the problem in this section. We take $H(x,v) = x - \ln(v), a = -1, b = 1, c = 1, d = 2$, and

$$g(v) = \exp\left(-v + \frac{p}{8(4+v^2)} + \frac{\arctan(v/2)}{16}\right),$$

$$h(v) = \exp\left(v + \frac{v}{8(4+v^2)} + \frac{\arctan(v/2)}{16}\right),$$

$$m(x) = \exp\left(x^3 + \frac{1}{40} + \frac{\arctan(1/2)}{16}\right),$$

$$n(x) = \exp\left(2x^3 + \frac{1}{32} + \frac{\arctan(1)}{16}\right).$$

In our problem $m = 2, n = 2$. We obtain the main reproducing kernel function as: $J_{k,p}(s,z) = A_k(s)B_p(z)$. where,

$$\begin{aligned}
A_k(s) =\ & \frac{7}{12} - \frac{5}{8}k - \frac{1}{24}k^3 + \frac{1}{4}k^2 - \frac{5}{8}s \\
& - \frac{3}{16}sk - \frac{1}{16}sk^3 \\
& + \frac{1}{4}s^2 + \frac{7}{8}s^2k + \frac{1}{8}s^2k^3 - \frac{3}{4}s^2k^2 \\
& - \frac{5}{24}s^3 - \frac{1}{16}s^3k \\
& + \frac{3}{8}sk^2 - \frac{1}{48}s^3k^3 + \frac{1}{8}s^3k^2, \quad s \le k,
\end{aligned}$$

$$\begin{aligned}
A_k(s) =\ & \frac{7}{12} - \frac{5}{8}s - \frac{1}{24}s^3 + \frac{1}{4}s^2 - \frac{5}{8}k \\
& - \frac{3}{16}ks - \frac{1}{16}ks^3 \\
& + \frac{1}{4}k^2 + \frac{7}{8}k^2s + \frac{1}{8}k^2s^3 - \frac{3}{4}k^2s^2 \\
& - \frac{5}{24}k^3 - \frac{1}{16}k^3s \\
& + \frac{3}{8}ks^2 - \frac{1}{48}k^3s^3 + \frac{1}{8}k^3s^2, \quad s > k,
\end{aligned}$$

$$\begin{aligned}
B_p(z) =\ & \frac{7}{12} - \frac{5}{8}p - \frac{5}{24}p^3 + \frac{1}{4}p^2 - \frac{5}{8}z \\
& - \frac{3}{16}zp - \frac{1}{16}zp^3 \\
& + \frac{7}{8}zp^2 + \frac{1}{4}z^2 + \frac{3}{8}z^2p + \frac{1}{8}z^2p^3 \\
& - \frac{3}{4}z^2p^2 - \frac{1}{24}z^3 \\
& - \frac{1}{16}z^3p - \frac{1}{48}z^3p^3 + \frac{1}{8}z^3p^2, \quad p \le z,
\end{aligned}$$

$$\begin{aligned}
B_p(z) =\ & \frac{7}{12} - \frac{5}{8}z - \frac{5}{24}z^3 + \frac{1}{4}z^2 - \frac{5}{8}p \\
& - \frac{3}{16}pz - \frac{1}{16}pz^3 \\
& + \frac{7}{8}pz^2 + \frac{1}{4}p^2 + \frac{3}{8}p^2z + \frac{1}{8}p^2z^3 \\
& - \frac{3}{4}p^2z^2 - \frac{1}{24}p^3 \\
& - \frac{1}{16}p^3z - \frac{1}{48}p^3z^3 + \frac{1}{8}p^3z^2, \quad z > p.
\end{aligned}$$

The reproducing kernel function $J_{(k,p)}(s,z)$ is in $W_2^{(2,2)}(\Omega)$.

Definition 7. *We say that a function $u : \Omega \to \mathbb{R}$ belongs to the binary space $W_2^{(1,1)}(\Omega)$ and write $u \in W_2^{(1,1)}(\Omega)$ provided $u \in AC(\Omega)$ and the following three square integrability conditions are satisfied [22]:*

(1) $u_x(\cdot, c) \in L^2[a,b]$;
(2) $u_t(a, \cdot) \in L^2[c,d]$;
(3) $u_{xt} \in L^2(\Omega)$.

Equip $W_2^{(1,1)}(\Omega)$ with the inner product

$$\begin{aligned}
\langle u, v \rangle_{W_2^{(1,1)}} =\ & u(a,c)v(a,c) + \int_a^b u_x(x,c)v_x(x,c)dx \\
& + \int_c^d u_t(a,t)v_t(a,t)dt \\
& + \int_c^d \int_a^b u_{xt}(x,t)v_{xt}(x,t)dxdt.
\end{aligned}$$

The binary space $W_2^{(1,1)}(\Omega)$ is a RKHS with reproducing kernel $G_{(k,p)}(s,z) = E_k(s)F_p(z)$.

4. Applications

The solution of (1)–(3) is given in the reproducing kernel space $W_2^{(2,2)}(\Omega)$ in this section. On defining the linear operator $N : W_2^{(2,2)}(\Omega) \to W_2^{(1,1)}(\Omega)$ by

$$\begin{aligned}
Nf =\ & M_v(x,v)f_x(x,v) - M_x(x,v)f_v(x,v) \\
& - \sigma(x,v)f, \quad f \in W_2^{(2,2)}(\Omega),
\end{aligned}$$

after homogenizing the boundary conditions, model problem (1)–(3) changes to the problem

$$\begin{aligned}
Nf =\ & H(x,v,f(x,v)), \\
& (x,v) \in [-1,1] \times [1,2], \\
& f(a,v) = f(b,v) = f(x,c) = f(x,d) = 0.
\end{aligned}$$

Lemma 3. *N is a bounded linear operator.*

Proof. Let $f \in W_2^{(2,2)}(\Omega)$ and $(x,v) \in \Omega$. We have

$$f(k,p) = \langle f, J_{(k,p)} \rangle_{W_2^{(2,2)}},$$

and

$$Nf(k,p) = \langle f, NJ_{(k,p)} \rangle_{W_2^{(2,2)}},$$

$$\frac{\partial}{\partial k}Nf(k,p) = \left\langle f, \frac{\partial}{\partial k}NJ_{(k,p)} \right\rangle_{W_2^{(2,2)}},$$

$$\frac{\partial}{\partial p}Nf(k,p) = \left\langle f, \frac{\partial}{\partial p}NJ_{(k,p)} \right\rangle_{W_2^{(2,2)}},$$

$$\frac{\partial}{\partial p}\frac{\partial}{\partial k}Nf(k,p) = \left\langle f, \frac{\partial}{\partial p}\frac{\partial}{\partial k}NJ_{(k,p)} \right\rangle_{W_2^{(2,2)}}.$$

Therefore, we have $a_0, b_0, a_1, b_1 > 0$ such that

$$|Nf(k,p)| \leq a_0 \|f\|_{W_2^{(2,2)}},$$

$$\left|\frac{\partial}{\partial p} Nf(k,p)\right| \leq b_0 \|f\|_{W_2^{(2,2)}},$$

$$\left|\frac{\partial}{\partial k} Nf(k,p)\right| \leq a_1 \|f\|_{W_2^{(2,2)}},$$

$$\left|\frac{\partial}{\partial p}\frac{\partial}{\partial k} Nf(k,p)\right| \leq b_1 \|f\|_{W_2^{(2,2)}}.$$

Thus, we get

$$
\begin{aligned}
\|Nf\|_{W_2^{(1,1)}}^2 &= \int_1^2 \left[\frac{\partial}{\partial p} Nf(-1,p)\right]^2 \mathrm{d}p \\
&+ \langle Nf(k,1), Nf(k,1)\rangle_{M_2^1} \\
&+ \int_{-1}^1 \int_1^2 \left[\frac{\partial}{\partial k}\frac{\partial}{\partial p} Nf(k,p)\right]^2 \mathrm{d}k\mathrm{d}p \\
&= \int_1^2 \left[\frac{\partial}{\partial p} Nf(-1,p)\right]^2 \mathrm{d}p \\
&+ [Nf(-1,-1)]^2 \\
&+ \int_{-1}^1 \left[\frac{\partial}{\partial k} Nf(k,1)\right]^2 \mathrm{d}k \\
&+ \int_{-1}^1 \int_1^2 \left[\frac{\partial}{\partial k}\frac{\partial}{\partial p} Nf(k,p)\right]^2 \mathrm{d}k\mathrm{d}p \\
&\leq (a_0^2 + a_1^2 + T(b_0^2 + b_1^2)) \|f\|_{W_2^{(2,2)}}.
\end{aligned}
$$

This completes the proof. $\qquad\square$

Now, choose a countable dense subset $\{(x_1,v_1),(x_2,v_2),\ldots\}$ in Ω and define

$$\varphi_i = G_{(x_i,v_i)}, \quad \Psi_i = L^*\varphi_i,$$

where N^* is the adjoint operator of N. The orthonormal system $\{\widehat{\Psi}_i\}_{i=1}^\infty$ of $W_2^{(2,2)}$ can be derived from the process of Gram–Schmidt orthogonalization of $\{\Psi_i\}_{i=1}^\infty$ as

$$\widehat{\Psi}_i = \sum_{k=1}^i \beta_{ik}\Psi_k.$$

Theorem 3. *Let us assume $\{(x_i,v_i)\}_{i=1}^\infty$ is dense in Ω. Then $\{\Psi_i(x,v)\}_{i=1}^\infty$ is a complete system in $W_2^{(2,2)}$, and*

$$\Psi_i = NJ_{(x_i,v_i)}(x,v).$$

Proof. We acquire

$$
\begin{aligned}
\Psi_i &= N^*\varphi_i = \langle N^*\varphi_i, J_{(x,v)}\rangle_{W_2^{(2,2)}} \\
&= \langle \varphi_i, NJ_{(x,v)}\rangle_{W_2^{(1,1)}} \\
&= \langle NJ_{(x,v)}, G_{(x_i,v_i)}\rangle_{W_2^{(1,1)}} \\
&= NJ_{(x,v)}(x_i,v_i) \\
&= NJ_{(x_i,v_i)}(x,v).
\end{aligned}
$$

It is obvious that $\Psi_i \in W_2^{(2,2)}$. For each fixed $f \in W_2^{(2,2)}$, if

$$\langle f, \Psi_i\rangle_{W_2^{(2,2)}} = 0, \quad i = 1,2,\ldots,$$

then

$$
\begin{aligned}
0 &= \langle f, \Psi_i\rangle_{W_2^{(1,1)}} \\
&= \langle f, N^*\varphi_i\rangle_{W_2^{(2,2)}} \\
&= \langle Nf, \varphi_i\rangle_{W_2^{(1,1)}} \\
&= Nf(x_i,v_i), \quad i = 1,2,\ldots.
\end{aligned}
$$

Note that $\{(x_i,v_i)\}_{i=1}^\infty$ is dense in Ω. Therefore, we obtain $Nf = 0$. From the existence of N^{-1}, it follows that $f = 0$. The proof is completed. $\quad\square$

Theorem 4. *If $\{(x_i,v_i)\}_{i=1}^\infty$ is dense in Ω, then the solution of the problem is obtained as:*

$$f(x,v) = \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} H(x_k,v_k,f(x_k,v_k))\widehat{\Psi}_i(x,v). \tag{6}$$

Proof. $\{\Psi_i(x,v)\}_{i=1}^\infty$ is a complete system in $W_2^{(2,2)}$. Therefore, we get

$$
\begin{aligned}
f &= \sum_{i=1}^\infty \langle f, \widehat{\Psi}_i\rangle_{W_2^{(2,2)}} \widehat{\Psi}_i \\
&= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle f, \Psi_k\rangle_{W_2^{(2,2)}} \widehat{\Psi}_i \\
&= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle f, N^*\varphi_k\rangle_{W_2^{(2,2)}} \widehat{\Psi}_i \\
&= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle Nf, \varphi_k\rangle_{W_2^{(1,1)}} \widehat{\Psi}_i \\
&= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle Nf, G_{(x_k,v_k)}\rangle_{W_2^{(1,1)}} \widehat{\Psi}_i \\
&= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} Nf(x_k,v_k)\widehat{\Psi}_i \\
&= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} H(x_k,v_k,f(x_k,v_k))\widehat{\Psi}_i(x,t).
\end{aligned}
$$

This completes the proof. $\qquad\square$

Now the approximate solution f_n can be obtained from the n-term intercept of the exact solution f and

$$f_n = \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} H(x_k,v_k,f(x_k,v_k))\widehat{\Psi}_i. \tag{7}$$

Obviously

$$\|f_n(x,v) - f(x,v)\|_{W_2^{(2,2)}} \to 0, \quad n \to \infty.$$

Theorem 5. *If $f \in W_2^{(2,2)}$ then, we have*

$$\|f_n(x,v) - f(x,v)\|_{W_2^{(2,2)}} \to 0, \quad n \to \infty.$$

Moreover a sequence $\|f_n(x,v) - f(x,v)\|_{W_2^{(2,2)}}$ is monotonically decreasing in n.

Proof. We have

$$\|f_n(x,v) - f(x,v)\|_{W_2^{(2,2)}} =$$

$$\left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} H(x_k, v_k, f(x_k,v_k)) \widehat{\Psi}_i(x,v) \right\|_{W_2^{(2,2)}}.$$

Therefore, we obtain

$$\|f_n(x,v) - f(x,v)\|_{W_2^{(2,2)}} \to 0, \quad n \to \infty.$$

Furthermore, we have

$$\|f_n(x,v) - f(x,v)\|_{W_2^{(2,2)}}^2 =$$

$$\left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} H(x_k, v_k, f(x_k,v_k)) \widehat{\Psi}_i(x,v) \right\|_{W_2^{(2,2)}}^2$$

$$= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} H(x_k, v_k, f(x_k,v_k)) \widehat{\Psi}_i(x,v) \right)^2.$$

It is obvious that $\|f_n(x,v) - f(x,v)\|_{W_2^{(2,2)}}$ is monotonically decreasing in n. □

To test the accuracy of the reproducing kernel Hilbert space mehod, an example has been given. The results are compared with the exact solutions. Let us take into consideration the problem of obtaining $(f(x,v), \sigma(x,v))$ in $\Omega = (-1,1) \times (1,2)$. The exact solution of the problem is given as [1]:

$$f(x,v) = \exp\left(x^3 v \frac{v}{8(4+v^2)} + \frac{\arctan(v/2)}{16} \right),$$

$$\sigma(x,v) = -3x^2 - x^3 - \frac{1}{(4+v^2)^2}.$$

Using our technique, we choose $25, 64$ and 100 points in the region $\Omega = [-1,1] \times [1,2]$ and obtain f_{25}, f_{64} and f_{100}. Numerical results are in good agreement with the exact solution. In order to prove the convergence of the exact solution we found absolute errors for different values of dense points n. We give the maximum absolute errors for different number of dense points in Table 1. The results demonstrate that the errors become smaller as n increases.

5. Conclusion

In this work, the reproducing kernel Hilbert space method was implemented for solving an inverse problem for the kinetic equation. Given technique is demonstrated to be of good convergence. It seems that this technique can also be applied to higher dimensional inverse problems. We found the reproducing kernel functions for solutions of a coefficient inverse problem for the kinetic equation. We concluded that these reproducing kernel functions can be used in much more complicated problems. We demonstrated our results by a table. These results proved the power of the reproducing kernel Hilbert space method.

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