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Exact boundary controllability of Galerkin approximations of Navier-Stokes system for soret convection

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Abstract. We study the exact controllability of finite dimensional Galerkin approximation of a Navier-Stokes type system describing doubly diffusive convection with Soret effect in a bounded smooth domain in $\mathbb{R}^d$ $(d = 2, 3)$ with controls on the boundary. The doubly diffusive convection system with Soret effect involves a difficult coupling through second order terms. The Galerkin approximations are introduced under certain assumptions on the Galerkin basis related to the linear independence of suitable traces of its elements over the boundary. By Using Hilbert uniqueness method in combination with a fixed point argument, we prove that the finite dimensional Galerkin approximations are exactly controllable.

Keywords: Exact boundary controllability; Galerkin approximation; doubly diffusive convection with Soret effect.

AMS Classification: 93B05,76R99,76D55,76D05.

1. Introduction

Control of fluid flows modeled by Navier-Stokes equations has received considerable attention due to its importance in practice and to the theoretical and computational challenges it poses. There is now an extensive body of literature devoted to this subject, see [13, 27, 11] for surveys in this area. In this paper, we consider controllability of a doubly diffusive convection with Soret effect modeled by a coupled Navier-Stokes type partial differential equations. The doubly diffusive system with Soret effect involves a difficult coupling through second order terms. Significant work has been devoted to studying the stability and physics of doubly diffusive convection with and without Soret effect (thermal diffusion), see for e.g. [3, 20, 26, 22, 24, 14]. These studies have reported convective flows lead to undesirable effects in certain applications. For example, thermosolutal convection is responsible for macrosegragation and can affect the uniformity and speed of growth rate in crystal growth. It is also responsible for erosion of gradient zone in solar ponds and roll-over instability (sudden over pressure) in storage and transport of gases. In spite of this, work concerning control of doubly diffusive flows is quite limited although there exists substantial work on control of thermal convection in fluid flows [15, 2], for example. In [28], control of temperature in doubly diffusive flows is studied computationally using boundary heat flux ignoring Soret effect. Optimal boundary control of doubly diffusive flows with Soret effect is studied in [21]. Mathematical aspects of doubly diffusive convection system such as existence and uniqueness can also be found in [21].

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The doubly diffusive system under study here includes as a particular case the classical incompressible Navier-Stokes equations. Therefore it is clear that one can not expect exact controllability of this system with arbitrary target functions due to its dissipative and non-reversibility properties. The approximate controllability, despite its questionable practical utility, has been addressed in [4] for the two dimensional Navier-Stokes equations in the iso-thermal and iso-concentration cases. However, the boundary conditions in that work are assumed to be non standard (slip boundary condition or the so called Navier slip boundary condition) and the problem of approximate controllability with classical Dirichlet boundary condition is still open. In [8, 12, 7] local exact controllability to uncontrolled trajectories of Navier-Stokes equations is proved. In [5], global exact controllability for the two-dimensional Navier-Stokes equations in a manifold without boundary is proved.

In [17, 18], exact controllability of finite dimensional Galerkin approximations of Navier-Stokes equations are proved. In the present work, we investigate the exact controllability for the doubly diffusive convection with Soret effect modeled by the Navier-Stokes system approached by Galerkin approximations.

The remainder of the paper is organized as follows. In Section 2, we present some preliminaries and study the wellposedness of the Navier-Stokes system for Soret Convection. In Section 3, we introduce Galerkin approximation of the doubly diffusive system and prove the exact boundary controllability result for this system. The proof uses the Hilbert uniqueness method to study the exact boundary controllability of linear system and a fixed point method.

2. Preliminaries

2.1. Notations

This section provides, for use in later sections, a summary of notations and function spaces. Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded domain with Lipschitzian boundary $\Gamma = \Gamma_D \cup \Gamma_N$. As usual, $L^p(\Omega)$, or simply L^p denotes the linear space of all real Lebesgue measurable functions ϕ and bounded in the usual norm denoted by $\|\phi\|_{L^p(\Omega)}$. The inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let $H^s(\Omega)$ be the usual Hilbertian Sobolev space with s derivatives in $L^2(\Omega)$. We denote with $\|\cdot\|_s$ the norm in $H^s(\Omega)$. The closed subspace of functions in $H^1(\Omega)$ with zero trace on Γ_D will be denoted by $H^1_D(\Omega)$. The closed subspace of functions in $L^2(\Omega)$ with zero mean on Ω will be denoted by $L^2_0(\Omega)$. The trace space $H^r(\Gamma)$ consists of functions that are the restriction to the boundary of functions in $H^{r+1/2}(\Omega)$, $r > 0$. We denote the norm and inner product for functions in $H^r(\Gamma)$ by $\|\cdot\|_{r,\Gamma}$ and $(\cdot,\cdot)_{r,\Gamma}$, respectively. In the sequel, we denote by boldface letters $\mathbb{R}^d$-valued function spaces such as $\mathbf{L}^p(\Omega) := [L^p(\Omega)]^d$ and $\mathbf{H}^r(\Omega) := [H^r(\Omega)]^d$. For details, see [1, 10]. We introduce the solenoidal spaces

$$\mathbf{V} := \left\{ \mathbf{v} \in \mathbf{H}^1_D(\Omega) \ : \quad \nabla \cdot \mathbf{v} = 0 \ \text{ in } \ \Omega \ \right\} ,$$

and

$$\mathbf{H} := \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ \nabla \cdot \mathbf{v} = 0 \text{ in } \ \Omega \, , \ (\mathbf{u} \cdot \mathbf{n})|_{\Gamma_D} = 0 \right\} .$$

We denote the dual of $\mathbf{V}$ by $\mathbf{V}^*$. If we identify $\mathbf{H}$ with its dual $\mathbf{H}^*$, then we get the following continuous and dense embedding:

$$\mathbf{V} \subset \mathbf{H} = \mathbf{H}^* \subset \mathbf{V}^* .$$

For a Banach space X, we denote by $L^p(0,T;X)$ the time-space function space endowed with the norm $\|w\|_{L^p(0,T;X)} := \left(\int_0^T \|w\|_X^p \, dt \right)^{1/p}$ if $1 \le p < \infty$ and $\text{esssup}_{t \in [0,T]} \|w\|_X$ if $p = \infty$. We will often use the abbreviated notation $L^p(X) := L^p(0,T;X)$ for convenience. We also introduce the space $\mathcal{W}(0,T) := \mathbf{W}_1 \times W_2 \times W_2$, where

$$\mathbf{W}_1 := L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$$

and

$$W_2 := L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_D(\Omega)) .$$

We end this section by recalling some inequalities that we will use in later sections.
Poincaré-Friedrichs' inequality: For $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$,

$$\lambda \|\mathbf{u}\|^2 \le \|\nabla \mathbf{u}\|^2 ,$$

where $\lambda > 0$ is a constant.
Young's inequality: For any $a, b \ge 0$ and $\epsilon > 0$, and $q, r > 1$

$$ab \le \frac{\epsilon}{q} a^q + \frac{\epsilon^{-\frac{r}{q}}}{r} b^r , \quad \text{with} \quad \frac{1}{q} + \frac{1}{r} = 1 .$$

2.2. Governing equations and weak formulation

In this section, we present the governing equations and study their well-posedness. The equations for the doubly diffusive convection with Soret effect in a binary mixture may be written, employing a Boussinesq approximation in the body force term in the momentum equation, as

$$\left.\begin{aligned} \partial_t \mathbf{u} &- Pr\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p \\ &= Pr^2 Gr_\theta(\theta + N_r S)\mathbf{i}_3 \,, \\ \nabla\cdot\mathbf{u} &= 0\,, \\ \partial_t\theta &+ \mathbf{u}\cdot\nabla\theta - \Delta\theta = 0\,, \\ \partial_t S &+ \mathbf{u}\cdot\nabla S - \tfrac{1}{Le}\Delta S = \tfrac{\alpha_*}{Le}\Delta\theta\,. \end{aligned}\right\} \quad (1)$$

with the boundary conditions

$$\mathbf{u}\,|_{\Gamma_D} = \mathbf{0}\,,\quad \theta\,|_{\Gamma_D} = 0\,,\quad S\,|_{\Gamma_D} = 0\,, \qquad (2)$$

$$\begin{aligned} [\mathbf{u} &+ \epsilon(-p\mathbf{n} + Pr\tfrac{\partial\mathbf{u}}{\partial\mathbf{n}} - \tfrac{1}{2}(\mathbf{u}\cdot\mathbf{n})\mathbf{u})]\,|_{\Gamma_N} = \mathbf{g}\,, \\ [\theta &+ \epsilon(\tfrac{\partial\theta}{\partial\mathbf{n}} - \tfrac{1}{2}(\mathbf{u}\cdot\mathbf{n})\theta)]\,|_{\Gamma_N} = h\,, \\ [S &+ \epsilon(\tfrac{1}{Le}\tfrac{\partial S}{\partial\mathbf{n}} - \tfrac{1}{2}(\mathbf{u}\cdot\mathbf{n})S)]\,|_{\Gamma_N} = f\,, \end{aligned} \qquad (3)$$

and initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})\,, \theta(\mathbf{x},0) = \theta_0(\mathbf{x})\,, S(\mathbf{x},0) = S_0(\mathbf{x}) \qquad (4)$$

in Ω where $\mathbf{u}$ is the velocity, θ the temperature, S the concentration and p the pressure. The non-dimensional parameters Pr, Le, Gr_θ and Gr_S denote the Prandtl number, the Lewis number, the thermal Grashof number and the species Grashof number, respectively. The ratio of species buoyancy to thermal buoyancy N_r is defined by $N_r = \frac{Gr_S}{Gr_\theta}$. In $(1)_4$ the first term on the right side corresponds to Soret effect. The cases $\alpha_* > 0$ and $\alpha_* < 0$ corresponds to positive and negative Soret effect, respectively. The Soret effects can have significant implications on convection in liquid mixtures, for example semi-conductor crystal growth [14]. Therefore the Dufour effect has been neglected here in comparison to Soret effect as is common for flows in liquid mixture. In addition, the constant ϵ in the boundary condition on Γ_N is non-negative. Note that by setting $\epsilon = 0$ the Robin boundary conditions become the Dirichlet boundary conditions. In fact, in actual computational implementation one can develop approximations of Dirichlet control problem by allowing $\epsilon \to 0^+$, see [21] for studies related to this in the context of optimal control.

Before proceeding, we present a little motivating discussion regarding the nonlinear Robin type boundary conditions. We observe that by integration by parts and $\mathbf{H}^{1/2}(\Gamma_N) \hookrightarrow \mathbf{L}^3(\Gamma_N)$ [1], the following identity holds for $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1_D(\Omega)$ with $\nabla\cdot\mathbf{u} = 0$:

$$((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) = -(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}) + (\mathbf{v},\mathbf{w}(\mathbf{u}\cdot\mathbf{n}))_{\Gamma_N}\,.$$

Similarly, we can show that

$$((\mathbf{u}\cdot\nabla)\theta,\phi) = -(\mathbf{u}\cdot\nabla\phi,\theta) + (\theta,\phi(\mathbf{u}\cdot\mathbf{n}))_{\Gamma_N}\,,$$

for $\theta,\phi \in H^1_D(\Omega), \mathbf{u} \in \mathbf{H}^1_D(\Omega)$ with $\nabla\cdot\mathbf{u} = 0$. If we define a tri-linear form $c(\cdot,\cdot,\cdot)$ as

$$\begin{aligned} c(\mathbf{u},\mathbf{w},\mathbf{v}) &:= \tfrac{1}{2}\,[((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}) - ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})] \\ &= (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}) - \tfrac{1}{2}(\mathbf{v},\mathbf{w}(\mathbf{u}\cdot\mathbf{n}))_{\Gamma_N} \end{aligned}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1_D(\Omega)$, it is clear that $c(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$. Similarly, if we define tri-linear forms $c_i(\mathbf{u},\chi,\phi),\ i = 1,2$ as

$$\begin{aligned} c_i(\mathbf{u},\chi,\phi) &:= \tfrac{1}{2}\,[((\mathbf{u}\cdot\nabla)\chi,\phi) - ((\mathbf{u}\cdot\nabla)\phi,\chi)] \\ &= (\mathbf{u}\cdot\nabla\chi,\phi) - \tfrac{1}{2}((\mathbf{u}\cdot\mathbf{n})\chi,\phi)_{\Gamma_N}\,, \end{aligned}$$

for $\chi,\phi \in H^1_D(\Omega), \mathbf{u} \in \mathbf{H}^1_D(\Omega)$, we can easily show that $c_i(\mathbf{u},\chi,\phi) = -c_i(\mathbf{u},\phi,\chi),\ i = 1,2\,.$

We now define the weak solution to the initial boundary value problem (1)–(4) as follows:

Definition 1. *Given $\mathbf{g} \in L^2(0,T;\mathbf{L}^2(\Gamma_N))$, $h,f \in L^2(0,T;L^2(\Gamma_N))$, a triple $(\mathbf{u},\theta,S) \in \mathcal{W}(0,T)$ is said to be a weak solution of (1)–(4) if*

$$\left.\begin{aligned} (\partial_t\mathbf{u},\mathbf{v}) &+ Pr(\nabla\mathbf{u},\nabla\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + \tfrac{1}{\epsilon}(\mathbf{u},\mathbf{v})_{\Gamma_N} \\ &= (Pr^2Gr_\theta(\theta + N_rS)\mathbf{i}_3,\mathbf{v}) + \tfrac{1}{\epsilon}(\mathbf{g},\mathbf{v})_{\Gamma_N}\,, \\ (\partial_t\theta,\phi) &+ (\nabla\theta,\nabla\phi) + c_1(\mathbf{u},\theta,\phi) + \tfrac{1}{\epsilon}(\theta,\phi)_{\Gamma_N} \\ &= \tfrac{1}{\epsilon}(h,\phi)_{\Gamma_N}\,, \\ (\partial_t S,\psi) &+ \tfrac{1}{Le}(\nabla S,\nabla\psi) + c_2(\mathbf{u},S,\psi) + \tfrac{1}{\epsilon}(S,\psi)_{\Gamma_N} \\ &= \tfrac{\alpha_*}{Le}(\nabla\theta,\nabla\psi) + \tfrac{1}{\epsilon}(f,\psi)_{\Gamma_N}\,, \end{aligned}\right\} \qquad (5)$$

and

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})\,, \theta(\mathbf{x},0) = \theta_0(\mathbf{x})\,, S(\mathbf{x},0) = S_0(\mathbf{x})\,,$$

for all $(\mathbf{v},\phi,\psi) \in \mathbf{V}\times(H^1_D(\Omega))^2$.

Proposition 1. *Assume $\mathbf{g} \in L^2(0,T;\mathbf{L}^2(\Gamma_N)), h \in L^2(0,T;L^2(\Gamma_N)), f \in L^2(0,T;L^2(\Gamma_N))$. Then, there exists a solution $(\mathbf{u},\theta,S) \in \mathcal{W}(0,T)$ satisfying (5) and*

$$\begin{aligned} \sup_{t\in[0,T]}\|\theta\|^2 &+ \|\nabla\theta\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \tfrac{1}{\epsilon}\|\theta\|_{L^2(0,T;L^2(\Gamma_N))} \le M_1\,, \\ \sup_{t\in[0,T]}\|S\|^2 &+ \tfrac{1}{Le}\|\nabla S\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \tfrac{1}{\epsilon}\|S\|_{L^2(0,T;L^2(\Gamma_N))} \le M_2\,, \end{aligned}$$

and

$$\begin{aligned} \sup_{t\in[0,T]}\|\mathbf{u}\|^2 &+ Pr\|\nabla\mathbf{u}\|^2_{L^2(0,T;\mathbf{H})} \\ &+ \tfrac{1}{\epsilon}\|\mathbf{u}\|^2_{L^2(0,T;L^2(\Gamma_N))} \le M_3 \\ &+ \tfrac{1}{\epsilon}\|\mathbf{g}\|^2_{L^2(0,T;L^2(\Gamma_N))} \end{aligned}$$

where $M_1 := \|\theta_0\|^2 + \frac{1}{\epsilon}\|h\|^2_{L^2(0,T;L^2(\Gamma_N))}$, $M_2 := \|S_0\|^2 + \frac{1}{\epsilon}\|f\|^2_{L^2(0,T;L^2(\Gamma_N)} + \frac{\alpha_*^2}{Le}M_1$ and $M_3 := \frac{Pr^3Gr^2}{\lambda}[M_1 + N_r^2M_2] + \|\mathbf{u}_0\|^2$.

Proof. We employ Galerkin approximation, a priori estimates and compactness methods to prove the existence of solutions. Let $\{(\mathbf{e}_k(\mathbf{x}),a_k(\mathbf{x}))\}_{k=1}^\infty$ be an orthogonal basis of $\mathbf{V}\times H^1_D(\Omega)$ such that $\{(\mathbf{e}_k(\mathbf{x}),a_k(\mathbf{x}))\}_{k=1}^\infty$ is linearly independent in $\mathbf{L}^2(\Gamma_N)\times L^2(\Gamma_N)$, see [19] for a proof of existence of such a basis. For each $m = 1,2,\ldots$, we set $\mathbf{V}^m := \text{span}\{\mathbf{e}_i\}_{i=1}^m \times (\text{span}\{a_i\}_{i=1}^m)^2$ and let

$\mathbf{u}_m = \sum_{k=1}^m c_k^{(m)}\mathbf{e}_k$, $\theta_m = \sum_{k=1}^m d_k^{(m)} a_k$ and $S_m = \sum_{k=1}^m r_k^{(m)} a_k$ be a solution of

$$\left.\begin{array}{rcl} (\partial_t \mathbf{u}_m, \mathbf{e}_k) &+& Pr(\nabla \mathbf{u}_m, \nabla \mathbf{e}_k) + c(\mathbf{u}_m, \mathbf{u}_m, \mathbf{e}_k) \\ &+& \frac{1}{\epsilon}(\mathbf{u}_m, \mathbf{e}_k)_{\Gamma_N} \\ &=& (Pr^2 Gr_\theta(\theta_m + N_r S_m)\mathbf{i}_3, \mathbf{e}_k) \\ &+& \frac{1}{\epsilon}(\mathbf{g}, \mathbf{e}_k)_{\Gamma_N}\,, \\ (\partial_t \theta_m, a_k) &+& (\nabla \theta_m, \nabla a_k) + c_1(\mathbf{u}_m, \theta_m, a_k) \\ &+& \frac{1}{\epsilon}(\theta_m, a_k)_{\Gamma_N} = \frac{1}{\epsilon}(h, a_k)_{\Gamma_N}\,, \\ (\partial_t S_m, a_k) &+& \frac{1}{Le}(\nabla S_m, \nabla a_k) + c_2(\mathbf{u}_m, S_m, a_k) \\ &+& \frac{1}{\epsilon}(S_m, a_k)_{\Gamma_N} \\ &=& \frac{\alpha_*}{Le}(\nabla \theta_m, \nabla a_k) + \frac{1}{\epsilon}(f, a_k)_{\Gamma_N}\,, \end{array}\right\} \tag{6}$$

$\mathbf{u}_m(0) = \mathbf{u}_{0m}$, $\theta_m(0) = \theta_{0m}$ and $S_m(0) = S_{0m}$, $k = 1, \ldots, m$, where $(\mathbf{u}_{0m}, \theta_{0m}, S_{0m})$ is the L^2- orthogonal projection of $(\mathbf{u}_0, \theta_0, S_0)$ onto the space $\mathbf{V}^m$. Since (6) is an initial value problem for nonlinear ODEs, existence of unique local solutions in some neighborhood $[0, t_m)$, for some $t_m > 0$, follows by Picard-Lindelöf theorem. The a-priori estimates we will prove later in $L^\infty(0,T;L^2(\Omega))$-norm show that continuation of solutions beyond t_m follows. We will employ energy methods to derive those a-priori estimates. First we multiply $(6)_1$ by $c_k^{(m)}$, $(6)_2$ by $d_k^{(m)}$ and $(6)_3$ by $r_k^{(m)}$, and add these equations for $k = 1, \ldots, m$. Using the skew-symmetry of the trilinear forms, we get

$$\left.\begin{array}{rcl} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}_m\|^2 &+& Pr\|\nabla \mathbf{u}_m\|^2 + \frac{1}{\epsilon}\|\mathbf{u}_m\|_{0,\Gamma_N}^2 \\ &=& (Pr^2 Gr_\theta(\theta_m + N_r S_m)\mathbf{i}_3, \mathbf{u}_m) \\ &+& \frac{1}{\epsilon}(\mathbf{g}, \mathbf{u}_m)_{\Gamma_N}\,, \\ \frac{1}{2}\frac{d}{dt}\|\theta_m\|^2 &+& \|\nabla \theta_m\|^2 + \frac{1}{\epsilon}\|\theta_m\|_{0,\Gamma_N}^2 \\ &=& \frac{1}{\epsilon}(h, \theta_m)_{\Gamma_N}\,, \\ \frac{1}{2}\frac{d}{dt}\|S_m\|^2 &+& \frac{1}{Le}\|\nabla S_m\|^2 + \frac{1}{\epsilon}\|S_m\|_{0,\Gamma_N}^2 \\ &=& \frac{\alpha_*}{Le}(\nabla \theta_m, \nabla S_m) \\ &+& \frac{1}{\epsilon}(f, S_m)_{\Gamma_N}\,. \end{array}\right\} \tag{7}$$

From the equation $(7)_2$, by integration we obtain the a priori estimate

$$\begin{array}{rcl} \sup_{t\in[0,T]}\|\theta_m(t)\|^2 &+& \|\theta_m\|_{L^2(0,T;H_D^1(\Omega))}^2 \\ &+& \frac{1}{\epsilon}\|\theta_m\|_{L^2(0,T;L^2(\Gamma_N))} \\ &\le& \|\theta_0\|^2 \\ &+& \frac{1}{\epsilon}\|h\|_{L^2(0,T;L^2(\Gamma_N))}^2 \\ &=& : M_1\,. \end{array} \tag{8}$$

By applying Cauchy-Schwarz inequality and Young's inequality in $(7)_3$, we obtain

$$\begin{array}{rcl} \frac{d}{dt}\|S_m\|^2 &+& \frac{1}{Le}\|\nabla S_m\|^2 + \frac{1}{\epsilon}\|S_m\|_{0,\Gamma_N}^2 \le \frac{\alpha_*^2}{Le}\|\nabla \theta_m\|^2 \\ &+& \frac{1}{\epsilon}\|f\|_{0,\Gamma_N}^2\,. \end{array}$$

Integrating this with respect time and using the fact that θ_m remains bounded in a bounded set of $L^2(0,T;H_D^1(\Omega))$, we obtain

$$\begin{array}{rcl} \sup_{t\in[0,T]}\|S_m(t)\|^2 &+& \frac{1}{Le}\|\nabla S_m\|_{L^2(L^2(\Omega))}^2 \\ &\le& \|S_0\|^2 \\ &+& \frac{1}{\epsilon}\|f\|_{L^2(L^2(\Gamma_N))} \\ &+& \frac{\alpha_*^2}{Le}M_1 =: M_2\,. \end{array} \tag{9}$$

Let us now turn to the a-priori estimate for $\mathbf{u}_m$. First we note that the right hand side of $(7)_1$ can be majorized using Young's inequality as follows

$$\begin{array}{rcl} (Pr^2 Gr_\theta(\theta_m &+& N_r S_m)\mathbf{i}_3, \mathbf{u}_m) + \frac{1}{\epsilon}(\mathbf{g}, \mathbf{u}_m)_{\Gamma_N} \\ &\le& \frac{Pr^3 Gr^2}{2\lambda}(\|\theta_m\|^2 + N_r^2\|S_m\|^2) \\ &+& \frac{\lambda Pr}{2}\|\mathbf{u}_m\|^2 \\ &+& \frac{1}{2\epsilon}\|\mathbf{g}\|_{0,\Gamma_N}^2 + \frac{1}{2\epsilon}\|\mathbf{u}_m\|_{0,\Gamma_N}^2\,. \end{array}$$

Applying the Poincare-Friedrichs inequality and employing the result in $(7)_1$ we obtain

$$\begin{array}{rcl} \frac{d}{dt}\|\mathbf{u}_m\|^2 &+& Pr\|\nabla \mathbf{u}_m\|^2 + \frac{1}{\epsilon}\|\mathbf{u}_m\|_{0,\Gamma_N}^2 \\ &\le& \frac{Pr^3 Gr^2}{\lambda}(\|\theta_m\|^2 + N_r^2\|S_m\|^2) + \frac{1}{\epsilon}\|\mathbf{g}\|_{0,\Gamma_N}^2\,. \end{array}$$

Integrating this with respect to time and using (8)–(9), we obtain

$$\begin{array}{rcl} \sup_{t\in[0,T]}\|\mathbf{u}_m\|^2 &+& Pr\|\nabla \mathbf{u}_m\|_{L^2(\mathbf{H})}^2 + \frac{1}{\epsilon}\|\mathbf{u}_m\|_{L^2(L^2(\Gamma_N))}^2 \\ &\le& M_3 + \frac{1}{\epsilon}\|\mathbf{g}\|_{L^2(L^2(\Gamma_N))}^2\,. \end{array} \tag{10}$$

In order to obtain bounds for $\partial_t \mathbf{u}_m$, we first notice that by Holder's inequality and the embedding $H_D^1(\Omega) \hookrightarrow L^p(\Omega)\,, p \le 6$, we have

$$\begin{array}{rcl} |c(\mathbf{u}, \mathbf{u}, \mathbf{v})| &\le& \frac{1}{2}\|\mathbf{u}\|_{L^3(\Omega)}\|\nabla \mathbf{u}\|\|\mathbf{v}\|_{L^6(\Omega)} \\ &+& \frac{1}{2}\|\mathbf{u}\|_{L^3(\Omega)}\|\nabla \mathbf{v}\|\|\mathbf{u}\|_{L^6(\Omega)} \\ &\le& C\|\mathbf{u}\|_1^2\|\mathbf{v}\|_1\,. \end{array}$$

From $(6)_1$, we obtain as usual

$$\begin{array}{rcl} \sup_{\mathbf{v}\in\mathbf{V}} \frac{|(\partial_t \mathbf{u}_m, \mathbf{v})|}{\|\mathbf{v}\|_1} &\le& Pr\|\nabla \mathbf{u}_m\| + C\|\mathbf{u}_m\|_1^2 \\ &+& Pr^2 Gr_\theta(\|\theta_m\| + N_r\|S_m\|) \\ &+& \frac{1}{\epsilon}\|\mathbf{g}\|_{L^2(\Gamma_N)} \\ &+& \frac{1}{\epsilon}\|\mathbf{u}_m\|_{L^2(\Gamma_N)} \end{array}$$

and thus $\|\partial_t \mathbf{u}_m\|_{L^1(\mathbf{V}^*)}$ is bounded due to the bounds in (8)–(10). Similarly, we can show $\|\partial_t \theta_m\|_{L^1(H_D^1(\Omega)^*)}$ and $\|\partial_t S_m\|_{L^1(H_D^1(\Omega)^*)}$ are bounded as well.

The a-priori estimates we obtained so far allow us to extract subsequences again denoted by $\{(\mathbf{u}_m, \theta_m, S_m)\}_{n=1}^\infty$ such that

$$\left.\mathbf{u}_m \to \mathbf{u}\quad\begin{array}{r} \text{weakly in } L^2(0,T;\mathbf{V}) \\ \text{weak star in } L^\infty(0,T;\mathbf{H}) \\ \text{strongly in } L^2(0,T;\mathbf{H}) \end{array}\right\}$$

and

$$\left.(\theta_m, S_m) \to (\theta, S)\quad\begin{array}{r} \text{weakly in } L^2(0,T;H_D^1(\Omega)) \\ \text{weak star in } L^\infty(0,T;L^2(\Omega)) \\ \text{strongly in } L^2(0,T;L^2(\Omega))\,. \end{array}\right\}$$

Here the strong convergence follows by the Aubin-Simon compactness lemma [25, Corollary 4, p.85] as we have the embeddings $\mathbf{V} \hookrightarrow\hookrightarrow \mathbf{H} \hookrightarrow \mathbf{V}^*$ and $H^1_D(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow H^1_D(\Omega)^*$.

Moreover, we have the following convergence results: $\mathbf{u}_m|_{\Gamma_N} \to \mathbf{u}|_{\Gamma_N}$ weakly in $L^2(0,T;\mathbf{L}^2(\Gamma_N))$, $\theta_m|_{\Gamma_N} \to \theta|_{\Gamma_N}$ weakly in $L^2(0,T;L^2(\Gamma_N))$ and $S_m|_{\Gamma_N} \to S|_{\Gamma_N}$ weakly in $L^2(0,T;L^2(\Gamma_N))$. When taking limit in (6), it is convenient for us to use the trilinear forms $c(\cdot,\cdot,\cdot)$ and $c_i(\cdot,\cdot,\cdot)$ involving boundary terms, i.e.,

$$c(\mathbf{u},\mathbf{u},\mathbf{v}) = -(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{u}) + \frac{1}{2}(\mathbf{v},\mathbf{u}(\mathbf{u}\cdot\mathbf{n}))_{\Gamma_N},$$

$$c_1(\mathbf{u},\theta,\phi) = -(\mathbf{u}\cdot\nabla\phi,\theta) + \frac{1}{2}(\theta,\phi(\mathbf{u}\cdot\mathbf{n}))_{\Gamma_N},$$

$$c_2(\mathbf{u},S,\phi) = -(\mathbf{u}\cdot\nabla S,\phi) + \frac{1}{2}(S,\phi(\mathbf{u}\cdot\mathbf{n}))_{\Gamma_N}.$$

However, the presence of nonlinear boundary terms require that we prove $\mathbf{u}_m\cdot\mathbf{n} \to \mathbf{u}\cdot\mathbf{n}$ in $L^2(0,T;L^2(\Gamma))$ strongly. In order to prove such a convergence, we first recall the integration by parts formula [10, Equation (I.2.17), p. 28]

$$\begin{aligned} < \mathbf{u}_m\cdot\mathbf{n} - \mathbf{u}\cdot\mathbf{n},\mathbf{v} >_{\Gamma_N} &= \int_\Omega \mathbf{v}\nabla\cdot(\mathbf{u}_m - \mathbf{u})\,d\Omega \\ &+ \int_\Omega (\mathbf{u}_m - \mathbf{u})\cdot\nabla\mathbf{v}\,d\Omega \end{aligned} \tag{11}$$

for all $\mathbf{v}\in\mathbf{H}^1(\Omega)$. By solving the variational problem

$$\left.\begin{aligned} (\partial_t\mathbf{v},\phi) + (\nabla\mathbf{v},\nabla\phi) &= 0 \quad \forall\phi\in H^1_0(\Omega) \\ \mathbf{v}|_{\Gamma_D} = \mathbf{0}, \qquad \mathbf{v}|_{\Gamma_N} &= \mathbf{u}\cdot\mathbf{n} - \mathbf{u}_m\cdot\mathbf{n} \\ \mathbf{v}(x,0) &= 0 \quad \text{in } \Omega \end{aligned}\right\} \tag{12}$$

we obtain a unique solution $\mathbf{v}\in L^2(0,T;\mathbf{H}^1(\Omega))$ such that

$$\|\nabla\mathbf{v}\|_{L^2(\mathbf{L}^2(\Omega))} \le C\|\mathbf{u}\cdot\mathbf{n} - \mathbf{u}_m\cdot\mathbf{n}\|_{L^2(\mathbf{H}^{1/2}(\Gamma_N))}.$$

Therefore by taking $\mathbf{v}$ in (11) to be the unique solution of this variational problem (12) and using the fact that $\nabla\cdot\mathbf{u}_m = \nabla\cdot\mathbf{u} = 0$ yields

$$\begin{aligned} \|\mathbf{u}_m\cdot\mathbf{n} - \mathbf{u}\cdot\mathbf{n}\|_{L^2(L^2(\Gamma_N))} &\le \|\mathbf{u}_m - \mathbf{u}\|_{L^2(\mathbf{L}^2(\Omega))} \\ &\quad \|\mathbf{u}_m\cdot\mathbf{n} - \mathbf{u}\cdot\mathbf{n}\|_{L^2(\mathbf{H}^{1/2}(\Gamma_N))}. \end{aligned} \tag{13}$$

The weak convergence $\mathbf{u}_m \to \mathbf{u}$ in $L^2(0,T;\mathbf{V})$ and trace theorem imply that $\mathbf{u}_m\cdot\mathbf{n} \to \mathbf{u}\cdot\mathbf{n}$ weakly in $L^2(0,T;\mathbf{H}^{\frac{1}{2}}(\Gamma_N))$ and thus $\|\mathbf{u}_m\cdot\mathbf{n} - \mathbf{u}\cdot\mathbf{n}\|_{L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))}$ is bounded. Therefore the required strong convergence follows from (11).

Let $\psi_i(t)$, $i=1,2,3$, be a continuously differentiable function on $[0,T]$ with $\psi_i(T)=0$. We multiply $(6)_i$ by ψ_i, $i=1,2,3$ and integrate with respect to time. Further, we integrate by parts in the time derivative term to move the derivative onto ψ_i. Now we can take limit in (6) by using standard techniques and show $(\mathbf{u},\theta,S)$ is indeed a solution of (5). The a-priori estimates in the lemma follow by taking the limit on the a-priori estimates (8)-(10) and using the weak lower semi-continuity of the norms. □

The uniqueness of the weak solutions discussed in Proposition 1 is an open problem. We denote by $(\mathbf{u}(\mathbf{x},t;(\mathbf{g},h,f)),\theta(\mathbf{x},t;(\mathbf{g},h,f)),S(\mathbf{x},t;(\mathbf{g},h,f)))$ the set of all possible solutions.

3. Exact controllability of Galerkin approximations

In this section, we introduce a Galerkin approximation of (1) and, for this finite dimensional system, we establish the exact controllability.

We consider finite dimensional spaces $\mathbf{E}_1 := \text{span}\{\mathbf{e}_1,\dots,\mathbf{e}_m\}$, $E_2 := \text{span}\{a_1,\dots,a_m\}$ and $E_3 := \text{span}\{a_1,\dots,a_m\}$ so that $\mathbf{E}_1 \subset \mathbf{V}$ and $E_i \subset H^1_D(\Omega)$, $i=2,3$. Recall the Galerkin approximation of the weak formulation (5) is

$$\left.\begin{aligned} (\partial_t\mathbf{u},\mathbf{e}_k) &+ Pr(\nabla\mathbf{u},\nabla\mathbf{e}_k) + c(\mathbf{u},\mathbf{u},\mathbf{e}_k) \\ &+ \tfrac{1}{\epsilon}(\mathbf{u},\mathbf{e}_k)_{\Gamma_N} \\ &= (Pr^2Gr_\theta(\theta+N_rS)\mathbf{i}_3,\mathbf{e}_k) \\ &+ \tfrac{1}{\epsilon}(\mathbf{g},\mathbf{e}_k)_{\Gamma_N}, \\ (\partial_t\theta,a_k) &+ (\nabla\theta,\nabla a_k) + c_1(\mathbf{u},\theta,a_k) + \tfrac{1}{\epsilon}(\theta,a_k)_{\Gamma_N} \\ &= \tfrac{1}{\epsilon}(h,a_k)_{\Gamma_N}, \\ (\partial_tS,a_k) &+ \tfrac{1}{Le}(\nabla S,\nabla a_k) + c_2(\mathbf{u},S,a_k) + \tfrac{1}{\epsilon}(S,a_k)_{\Gamma_N} \\ &= \tfrac{\alpha_*}{Le}(\nabla\theta,\nabla a_k) \\ &+ \tfrac{1}{\epsilon}(f,a_k)_{\Gamma_N} \end{aligned}\right\} \tag{14}$$

and $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 \in \mathbf{E}_1$, $\theta(\mathbf{x},0) = \theta_0 \in E_2$ and $S(\mathbf{x},0) = S_0 \in E_3$, for $k=1,2,\dots,m$. The system (14) has a unique solution $(\mathbf{u},\theta,S)\in C(0,T;\mathbf{E}_1)\times C(0,T;E_2)\times C(0,T;E_3)$ for any $T>0$.

Definition 2. *The Galerkin approximation (14) is said to be exactly controllable at time $T>0$ if, for any given $(\mathbf{u}_0,\theta_0,S_0),(\mathbf{u}_T,\theta_T,S_T)\in\mathbf{E}_1\times E_2\times E_3$, there exists controls $(\mathbf{g},h,f)\in\mathbf{L}^2((0,T)\times\Gamma_N)\times L^2((0,T)\times\Gamma_N)\times L^2((0,T)\times\Gamma_N)$ such that the solution $(\mathbf{u},\theta,S)$ of (14) satisfies*

$$(\mathbf{u}(\cdot,T;\widehat{\mathbf{g}}),\theta(\cdot,T;\widehat{\mathbf{g}}),S(\cdot,T;\widehat{\mathbf{g}})) = (\mathbf{u}_T,\theta_T,S_T), \tag{15}$$

where $\widehat{\mathbf{g}} := (\mathbf{g},h,f)$.

Let

$$\mathcal{J}(\mathbf{g},h,f) = \frac{1}{2}\int_{\Gamma_N\times(0,T)} |\mathbf{g}|^2 + |h|^2 + |f|^2\,dx\,dt$$

be the cost to achieve (15). The main result of this section is as follows.

Theorem 1. *The Galerkin approximation (14) is exactly controllable in the sense of (15). Moreover, the cost of control $\mathcal{J}(\mathbf{g},h,f)$ is bounded independently of the nonlinearity.*

Proof. The proof of this theorem uses a fixed point argument. In order to show and make explicit that the cost of control can be bounded independent of nonlinearity, we introduce a family of state equations

$$\left.\begin{aligned} \partial_t\mathbf{u} - Pr\Delta\mathbf{u} &+ \alpha(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p \\ &= Pr^2Gr_\theta(\theta+N_rS)\mathbf{i}_3, \\ \nabla\cdot\mathbf{u} &= 0, \\ \partial_t\theta - \Delta\theta &+ \beta(\mathbf{u}\cdot\nabla)\theta = 0, \\ \partial_tS - \tfrac{1}{Le}\Delta S &+ \gamma(\mathbf{u}\cdot\nabla)S = \tfrac{\alpha_*}{Le}\Delta\theta. \end{aligned}\right\} \tag{16}$$

with the boundary conditions

$$\mathbf{u}\,|_{\Gamma_D} = \mathbf{0}\,,\quad \theta\,|_{\Gamma_D} = 0\,,\quad S\,|_{\Gamma_D} = 0\,,$$
$$[\epsilon(-p\mathbf{n} + Pr\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - \frac{\alpha}{2}(\mathbf{u}\cdot\mathbf{n})\mathbf{u}) + \mathbf{u}]\,|_{\Gamma_N} = \mathbf{g}\,,$$
$$[\theta + \epsilon(\frac{\partial \theta}{\partial \mathbf{n}} - \frac{\beta}{2}(\mathbf{u}\cdot\mathbf{n})\theta]\,|_{\Gamma_N} = h\,,$$
$$[S + \epsilon(\frac{1}{Le}\frac{\partial S}{\partial \mathbf{n}} - \frac{\gamma}{2}(\mathbf{u}\cdot\mathbf{n})S]\,|_{\Gamma_N} = f\,,$$

and initial conditions $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$, $\theta(\mathbf{x},0) = \theta_0(\mathbf{x})$, and $S(\mathbf{x},0) = S_0(\mathbf{x})$ in Ω, where $\alpha,\beta,\gamma \in \mathbb{R}$. A Galerkin approximation of (16) is

$$\left.\begin{array}{rcl}
(\partial_t \mathbf{u}, \mathbf{e}_k) & + & Pr(\nabla \mathbf{u}, \nabla \mathbf{e}_k) + \alpha c(\mathbf{u},\mathbf{u},\mathbf{e}_k) \\
 & + & \frac{1}{\epsilon}(\mathbf{u},\mathbf{e}_k)_{\Gamma_N} \\
 & = & (Pr^2 Gr_\theta(\theta + N_r S)\mathbf{i}_3, \mathbf{e}_k) \\
 & + & \frac{1}{\epsilon}(\mathbf{g},\mathbf{e}_k)_{\Gamma_N}\,, \\
(\partial_t \theta, a_k) & + & (\nabla\theta, \nabla a_k) + \beta c_1(\mathbf{u},\theta,a_k) \\
 & + & \frac{1}{\epsilon}(\theta, a_k)_{\Gamma_N} = \frac{1}{\epsilon}(h,a_k)_{\Gamma_N}\,, \\
(\partial_t S, a_k) & + & \frac{1}{Le}(\nabla S, \nabla a_k) + \gamma c_2(\mathbf{u},S,a_k) \\
 & + & \frac{1}{\epsilon}(S,a_k)_{\Gamma_N} \\
 & = & \frac{\alpha_*}{Le}(\nabla\theta,\nabla a_k) + \frac{1}{\epsilon}(f,a_k)_{\Gamma_N}\,,
\end{array}\right\} \tag{17}$$

and $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 \in \mathbf{E}_1$, $\theta(\mathbf{x},0) = \theta_0 \in E_2$ and $S(\mathbf{x},0) = S_0 \in E_3$, for $k = 1,2,\ldots,m$. Given $\mathbf{U} \in L^2(0,T;\mathbf{E}_1)$, we analyze the exact controllability of the linearized system

$$\left.\begin{array}{rcl}
(\partial_t \mathbf{u}, \mathbf{e}_k) & + & Pr(\nabla \mathbf{u}, \nabla \mathbf{e}_k) + \alpha c(\mathbf{U},\mathbf{u},\mathbf{e}_k) \\
 & + & \frac{1}{\epsilon}(\mathbf{u},\mathbf{e}_k)_{\Gamma_N} \\
 & = & (Pr^2 Gr_\theta(\theta + N_r S)\mathbf{i}_3, \mathbf{e}_k) \\
 & + & \frac{1}{\epsilon}(\mathbf{g},\mathbf{e}_k)_{\Gamma_N}\,, \\
(\partial_t \theta, a_k) & + & (\nabla\theta, \nabla a_k) + \beta c_1(\mathbf{U},\theta,a_k) \\
 & + & \frac{1}{\epsilon}(\theta, a_k)_{\Gamma_N} = \frac{1}{\epsilon}(h,a_k)_{\Gamma_N}\,, \\
(\partial_t S, a_k) & + & \frac{1}{Le}(\nabla S, \nabla a_k) + \gamma c_2(\mathbf{U},S,a_k) \\
 & + & \frac{1}{\epsilon}(S,a_k)_{\Gamma_N} \\
 & = & \frac{\alpha_*}{Le}(\nabla\theta,\nabla a_k) + \frac{1}{\epsilon}(f,a_k)_{\Gamma_N}\,,
\end{array}\right\} \tag{18}$$

and

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0}\,,\quad \theta(\mathbf{x},0) = 0\,,\quad \text{and}\quad S(\mathbf{x},0) = 0\,,$$

for $k = 1,2,\ldots,m$. We prove the exact controllability of system (18) using the Hilbert Uniqueness Method [16]. Notice that we have set the initial conditions to zero. However, due to the linearity, all results are valid as well if the initial data is not zero, i.e., $\mathbf{u}(0) = \mathbf{u}_0 \in \mathbf{E}_1$, $\theta(0) = \theta_0 \in E_2$ and $S(0) = S_0 \in E_3$. We will show that the system (18) is exactly controllable in time $T > 0$. For this, it is enough to show that if $(\mathbf{g}_1, g_2, g_3) \in \mathbf{E}_1 \times E_2 \times E_3$ satisfies

$$\begin{array}{rcl}
((\mathbf{g}_1,g_2,g_3),(\mathbf{u}(\cdot,T;\widehat{\mathbf{g}}),\theta(\cdot,T;\widehat{\mathbf{g}}),S(\cdot,T;\widehat{\mathbf{g}})) & = & 0 \\
\forall(\widehat{\mathbf{g}},h,f) \in L^2((0,T)\times\Gamma_N)^3 & & \\
\text{then } \widehat{\mathbf{g}} \equiv \mathbf{0}\,, & &
\end{array} \tag{19}$$

where $\widehat{\mathbf{g}} := (\mathbf{g},h,f)$. Let $(\boldsymbol{\mu},\zeta,\xi)$ be the solution to the adjoint system

$$\left.\begin{array}{rcl}
-(\partial_t \boldsymbol{\mu}, \mathbf{e}_k) & + & Pr(\nabla\boldsymbol{\mu},\nabla\mathbf{e}_k) - \alpha c(\mathbf{U},\boldsymbol{\mu},\mathbf{e}_k) \\
 & + & \frac{1}{\epsilon}(\boldsymbol{\mu},\mathbf{e}_k)_{\Gamma_N} = 0 \\
-(\partial_t \zeta, a_k) & + & (\nabla\zeta,\nabla a_k) - \beta c_1(\mathbf{U},\zeta,a_k) \\
 & + & \frac{1}{\epsilon}(\zeta,a_k)_{\Gamma_N} \\
 & = & (Pr^2 Gr_\theta \boldsymbol{\mu}\mathbf{i}_3, a_k) + \frac{\alpha_*}{Le}(\nabla\xi,\nabla a_k)\,, \\
-(\partial_t \xi, a_k) & + & \frac{1}{Le}(\nabla\xi,\nabla a_k) - \gamma c_2(\mathbf{U},\xi,a_k) \\
 & + & \frac{1}{\epsilon}(\xi,a_k)_{\Gamma_N} \\
 & = & (Pr^2 Gr_\theta N_r \boldsymbol{\mu}\mathbf{i}_3, a_k)\,,
\end{array}\right\} \tag{20}$$

and

$$\boldsymbol{\mu}(T) = \mathbf{g}_1\,,\quad \zeta(T) = g_2\,,\quad \text{and}\quad \xi(T) = g_3\,,$$

for $k = 1,2,\ldots,m$. This system clearly has a unique solution $(\boldsymbol{\mu},\zeta,\xi) \in C(0,T;\mathbf{E}_1)\times C(0,T;E_2)\times C(0,T;E_3)$. It follows from (20) by integration with respect to time and integration by parts that

$$\left.\begin{array}{rcl}
-(\mathbf{u}(T),\boldsymbol{\mu}(T)) & + & \int_0^T (\partial_t \mathbf{u},\boldsymbol{\mu})\,dt \\
 & + & \int_0^T Pr(\nabla\boldsymbol{\mu},\nabla\mathbf{u})\,dt \\
 & + & \alpha\int_0^T c(\mathbf{U},\mathbf{u},\boldsymbol{\mu})\,dt \\
 & + & \frac{1}{\epsilon}\int_0^T (\boldsymbol{\mu},\mathbf{u})_{\Gamma_N}\,dt = 0 \\
-(\zeta(T),\theta(T)) & + & \int_0^T (\partial_t\theta,\zeta)\,dt \\
 & + & \int_0^T (\nabla\zeta,\nabla\theta)\,dt \\
 & + & \int_0^T \beta c_1(\mathbf{U},\theta,\zeta)\,dt \\
 & + & \frac{1}{\epsilon}\int_0^T (\zeta,a_k)_{\Gamma_N}\,dt \\
 & = & \int_0^T (Pr^2 Gr_\theta\boldsymbol{\mu}\mathbf{i}_3,\theta)\,dt \\
 & + & \frac{\alpha_*}{Le}(\nabla\xi,\nabla\theta)\,dt \\
-(\xi(T),S(T)) & + & \int_0^T (\partial_t S,\xi)\,dt \\
 & + & \frac{1}{Le}\int_0^T (\nabla\xi,\nabla S)\,dt \\
 & + & \gamma\int_0^T c_2(\mathbf{U},S,\xi)\,dt \\
 & + & \frac{1}{\epsilon}\int_0^T (\xi,S)_{\Gamma_N}\,dt \\
 & = & \int_0^T (Pr^2 Gr_\theta N_r\boldsymbol{\mu}\mathbf{i}_3,S)\,dt\,.
\end{array}\right\} \tag{21}$$

Adding $(21)_1 - (21)_3$ and using (18) yields

$$\begin{array}{rcl}
(\mathbf{u}(T),\mathbf{g}_1) & + & (\theta(T),g_2) + (S(T),g_3) \\
 & = & \frac{1}{\epsilon}\int_0^T (\boldsymbol{\mu},\mathbf{g})_{\Gamma_N}\,dt + \frac{1}{\epsilon}\int_0^T (h,\zeta)_{\Gamma_N}\,dt \\
 & + & \frac{1}{\epsilon}\int_0^T (f,\xi)_{\Gamma_N}\,dt\,.
\end{array} \tag{22}$$

If (19) holds, then by (22) we have that

$$\begin{array}{rcl}
\frac{1}{\epsilon}\int_0^T (\boldsymbol{\mu},\mathbf{g})_{\Gamma_N}\,dt & + & \frac{1}{\epsilon}\int_0^T (h,\zeta)_{\Gamma_N}\,dt \\
 & + & \frac{1}{\epsilon}\int_0^T (f,\xi)_{\Gamma_N}\,dt \\
 & = & 0
\end{array}$$

for all $(\mathbf{g},h,f) \in L^2((0,T)\times\Gamma_N)^3$. This yields $(\boldsymbol{\mu},\zeta,\xi) = 0$ on $\Gamma_N \times (0,T)$. But $(\boldsymbol{\mu},\zeta,\xi) =$

$\sum_{i=1}^{n}(\boldsymbol{\mu}_i(t)\mathbf{e}_i, \zeta_i(t)a_i, \xi_i(t)a_i)$ and by the assumption on the basis $\{(\mathbf{e}_i, a_i)\}_{i=1}^{n}$, it follows that $(\boldsymbol{\mu}_i(t), \zeta_i(t), \xi_i(t)) = (0,0,0)$ for $i = 1, 2, \dots, n$. Hence $(\boldsymbol{\mu}, \zeta, \xi) \equiv (\mathbf{0}, 0, 0)$ and, thus $(\mathbf{g}_1, g_2, g_3) \equiv 0$. Therefore the linear system (18) is exactly controllable.

The result we just obtained allows us to define a functional $G : L^2(0,T;\mathbf{E}) \to \mathbb{R}$ by

$$G(U) = \inf_{(\mathbf{g},h,f)\in\mathcal{U}_{ad}} \frac{1}{2} \int_{\Gamma_N\times(0,T)} |\mathbf{g}|^2 + |h|^2 + |f|^2\, dx\, dt\,,$$

where

$$\mathcal{U}_{ad} := \{(\mathbf{g}, h, f) \in L^2(\Gamma_N \times (0,T))^3 \ : (\mathbf{u}, \theta, S) \text{ is the solution to (18) that satisfies (15)}\}$$

is the set of admissible controls. We will use a duality argument to prove that $G(U)$ is bounded by a constant independent of U, α, β and γ. That is,

$$G(U) \leq C\,, \quad \text{where C is a constant independent of U},\alpha\,\beta \text{ and } \gamma\,. \tag{23}$$

Let $\mathcal{L} : L^2(\Gamma_N \times (0,T))^3 \to \mathbf{E}_1 \times E_2 \times E_3$ be a linear continuous operator defined by

$$\mathcal{L}(\widehat{\mathbf{g}}) = (\mathbf{u}(\cdot,T;\widehat{\mathbf{g}}), \theta(\cdot,T;\widehat{\mathbf{g}}), S(\cdot,T;\widehat{\mathbf{g}}))\,,$$

where $\widehat{g} := (\mathbf{g}, h, f)$.

Let us also define two functionals $F_1(\mathbf{g}, h, f)$ and $F_2(\mathbf{g}_1, g_2, g_3)$ by

$$F_1(\mathbf{g}, h, f) = \frac{1}{2} \int_{\Gamma_N\times(0,T)} |\mathbf{g}|^2 + |h|^2 + |f|^2\, dx\, dt$$

and

$$F_2(\mathbf{g}_1, g_2, g_3) = \begin{cases} 0\,, & \text{if } (\mathbf{g}_1, g_2, g_3) = (\mathbf{u}_T, \theta_T, S_T) \\ \infty\,, & \text{otherwise}\,. \end{cases}$$

Then the functional $G(U)$ can be written as

$$G(U) = \inf_{(\mathbf{g},h,f)\in L^2(\Gamma_N\times(0,T))^3} [F_1(\mathbf{g},h,f) + F_2(\mathcal{L}(\mathbf{g},h,f))]$$

and by the duality theorem of Fenchel and Rockafellar, see for example [23], we have that

$$-G(U) = \inf_{\breve{\mathbf{g}}\in L^2(\Gamma_N\times(0,T))^3} [F_1^*(\mathcal{L}^*(\breve{\mathbf{g}})) + F_2(-\breve{\mathbf{g}})]\,, \tag{24}$$

where $\breve{\mathbf{g}} := (\mathbf{g}_1, g_2, g_3)$ and $\mathcal{L}^* : \mathbf{E}_1 \times E_2 \times E_3 \to L^2(\Gamma_N \times (0,T))^3$ is the adjoint of $\mathcal{L}$, and F_1^* and F_2^* are the Fenchel conjugate of F_1 and F_2, respectively. It follows easily from (22) that the adjoint $\mathcal{L}^*$ is given by

$$\mathcal{L}^*(\mathbf{g}_1, g_2, g_3) = (\boldsymbol{\mu}, \zeta, \xi) \quad \text{in} \quad \Gamma_N \times (0,T)\,.$$

Moreover, the Fenchel conjugate F_1^* and F_2^* can be shown to be given by

$$F_1^*(\boldsymbol{\mu}, \zeta, \xi) = \frac{1}{2} \int_{\Gamma_N\times(0,T)} |\boldsymbol{\mu}|^2 + |\zeta|^2 + |\xi|^2\, dx\, dt$$

and

$$F_2^*(-(\mathbf{g}_1, g_2, g_3)) = -((\mathbf{g}_1, g_2, g_3), (\mathbf{u}_T, \theta_T, S_T))\,.$$

Therefore (24) becomes

$$\begin{aligned} -G(U) &= \inf_{\breve{\mathbf{g}}\in\Pi_{i=1}^3 E_i}[\int_{\Gamma_N\times(0,T)} \tfrac{1}{2}(|\boldsymbol{\mu}|^2 + |\zeta|^2 + |\xi|^2) dx dt \\ &- (\breve{\mathbf{g}}, (\mathbf{u}_T, \theta_T, S_T))]\,, \end{aligned} \tag{25}$$

where $\breve{\mathbf{g}} := (\mathbf{g}_1, g_2, g_3)$. But, in view of the assumptions on the bases for $\mathbf{E}_1$, E_2 and E_3, $\int_{\Gamma_N} |\mathbf{e}_1|^2 + |e_2|^2 + |e_3|^2\, ds$ is a norm on $\mathbf{E}_1 \times E_2 \times E_3$ so that

$$\begin{aligned} c_1\|(\mathbf{e}_1, e_2, e_3)\|^2 &\leq \int_{\Gamma_N} |\mathbf{e}_1|^2 + |e_2|^2 + |e_3|^2\, ds \\ &\leq c_2\|(\mathbf{e}_1, e_2, e_3)\|^2 \\ &\forall \quad (\mathbf{e}_1, e_2, e_3) \in \mathbf{E}_1 \times E_2 \times E_3 \end{aligned}$$

for some constants c_1 and c_2 depending only on $\mathbf{E}_1$, E_2 and E_3. Therefore (25) can be written as

$$\begin{aligned} -G(U) &= \inf_{\breve{\mathbf{g}}\in\Pi_{i=1}^3 \mathbf{E}_i} \Bigg[\tfrac{c_1}{2} \int_{\Omega\times(0,T)} |\boldsymbol{\mu}|^2 + |\zeta|^2 \\ &+ |\xi|^2\, dx\, dt - (\breve{\mathbf{g}}, (\mathbf{u}_T, \theta_T, S_T))\Bigg] \end{aligned} \tag{26}$$

where $\breve{\mathbf{g}} := (\mathbf{g}_1, g_2, g_3)$. From (20), we obtain by the skew symmetry properties of the trilinear forms that

$$-\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\mu}\|^2 + Pr\|\nabla\boldsymbol{\mu}\|^2 + \frac{1}{\epsilon}\|\boldsymbol{\mu}\|_{0,\Gamma_N}^2 = 0\,,$$

$$\begin{aligned} -\tfrac{1}{2}\tfrac{d}{dt}\|\zeta\|^2 &+ \|\nabla\zeta\|^2 + \tfrac{1}{\epsilon}\|\zeta\|_{0,\Gamma_N}^2 \\ &= (Pr^2 Gr_\theta \boldsymbol{\mu}\mathbf{i}_3, \zeta) \\ &+ \tfrac{\alpha_*}{Le}(\nabla\xi, \nabla\zeta) \end{aligned}$$

$$\begin{aligned} -\tfrac{1}{2}\tfrac{d}{dt}\|\xi\|^2 &+ \tfrac{1}{Le}\|\nabla\xi\|^2 + \tfrac{1}{\epsilon}\|\xi\|_{0,\Gamma_N}^2 \\ &= (Pr^2 N_r Gr_\theta \boldsymbol{\mu}\mathbf{i}_3, \xi)\,, \end{aligned}$$

$$\boldsymbol{\mu}(T) = \mathbf{g}_1\,, \quad \zeta(T) = g_2\,, \quad \xi(T) = g_3\,.$$

Integrating the preceding differential equations with respect to time from t to T yields

$$\begin{aligned} \tfrac{1}{2}\|\boldsymbol{\mu}\|^2 &+ Pr\int_t^T \|\nabla\boldsymbol{\mu}\|^2\, dt + \tfrac{1}{\epsilon}\int_t^T \|\boldsymbol{\mu}\|_{0,\Gamma_N}^2\, dt \\ &= \tfrac{1}{2}\|\mathbf{g}_1\|^2\,, \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}\|\zeta\|^2 &+ \int_t^T \|\nabla\zeta\|^2\, dt + \tfrac{1}{\epsilon}\int_t^T \|\zeta\|_{0,\Gamma_N}^2\, dt \\ &= \int_t^T (Pr^2 Gr_\theta \boldsymbol{\mu}\mathbf{i}_3, \zeta)\, dt \\ &+ \tfrac{\alpha_*}{Le}\int_t^T (\nabla\xi, \nabla\zeta)\, dt \\ &+ \tfrac{1}{2}\|g_2\|^2 \end{aligned}$$

and

$$\begin{aligned} \tfrac{1}{2}\|\xi\|^2 &+ \tfrac{1}{Le}\int_t^T \|\nabla\xi\|^2\, dt + \tfrac{1}{\epsilon}\int_t^T \|\xi\|_{0,\Gamma_N}^2\, dt \\ &= \int_t^T (Pr^2 N_r Gr_\theta \boldsymbol{\mu}\mathbf{i}_3, \xi)\, ds + \tfrac{1}{2}\|g_3\|^2\,. \end{aligned}$$

Adding the last three equations and integrating with respect to t from 0 to T yields

$$\begin{aligned}\frac{1}{2}\int_0^T \|\boldsymbol{\mu}\|^2 &+ \|\zeta\|^2+\|\xi\|^2\,dt+\int_0^T t\,[Pr\|\nabla\boldsymbol{\mu}\|^2\\ &+ \|\nabla\zeta\|^2+\tfrac{1}{Le}\|\nabla\xi\|^2]\,ds\\ &+ \tfrac{1}{\epsilon}\int_0^T t\,[\|\boldsymbol{\mu}\|^2_{0,\Gamma_N}+\|\zeta\|^2_{0,\Gamma_N}\\ &+ \|\xi\|^2_{0,\Gamma_N}]\,ds\\ &= \tfrac{T}{2}(\|\mathbf{g}_1\|^2+\|g_2\|^2+\|g_3\|^2)\\ &+ \int_0^T t\,[(Pr^2Gr_\theta\boldsymbol{\mu}\mathbf{i}_3,\zeta)+\tfrac{\alpha_*}{Le}(\nabla\xi,\nabla\zeta)\\ &+ (Pr^2Gr_\theta N_r\boldsymbol{\mu}\mathbf{i}_3,\xi)]\,ds\,.\end{aligned} \tag{27}$$

Notice by the finite dimensionality of the spaces $\mathbf{E}_1, E_2$ and E_3, we have

$$\begin{aligned} Pr\|\nabla\boldsymbol{\mu}\|^2+\|\nabla\zeta\|^2+\tfrac{1}{Le}\|\nabla\xi\|^2 &\le C\|(\boldsymbol{\mu},\zeta,\xi)\|^2\\ \int_\Omega \nabla\xi\cdot\nabla\zeta\,dx &\le C\|(\xi,\zeta)\|^2\,,\end{aligned} \tag{28}$$

for some constant $C>0$ that depends only on $\mathbf{E}_1$, E_2 and E_3 are finite dimensional. Also by the finite dimensionality of $\mathbf{E}_1$, E_2 and E_3, we have

$$\|\boldsymbol{\mu}\|^2_{0,\Gamma_N}+\|\zeta\|^2_{0,\Gamma_N}+\|\xi\|^2_{0,\Gamma_N}\le C\|\boldsymbol{\mu}\|^2+\|\zeta\|^2+\|\xi\|^2\,. \tag{29}$$

for some constant $C>0$ depending only on $\mathbf{E}_1$, E_2 and E_3. Using (28) and (29) in (27) yields

$$\begin{aligned}\tfrac{T}{2}(\|\mathbf{g}_1\|^2 &+ \|g_2\|^2+\|g_3\|^2)\le(\tfrac{1}{2}+CT(1+\tfrac{1}{\epsilon}))\\ &\int_0^T\|\boldsymbol{\mu}\|^2+\|\zeta\|^2+\|\xi\|^2\,ds\,,\end{aligned} \tag{30}$$

for some constant depending only on $\mathbf{E}_1$, E_2 and E_3, $Pr, Le, \alpha_*, Gr_\theta$ and Nr. Employing (30) in (26), we have

$$\begin{aligned}-G(U) &\ge \inf_{\check{\mathbf{g}}\in\Pi_{i=1}^3\mathbf{E}_i}\left[\frac{c_1T}{2(1+2CT(1+\frac{1}{\epsilon}))}\right.\\ &(\|\mathbf{g}_1\|^2+\|g_2\|^2+\|g_3\|^2)\\ &- \left.(\check{\mathbf{g}}_1,(\mathbf{u}_T,\theta_T,S_T))\right]\,,\end{aligned}$$

where $\check{\mathbf{g}}:=(\mathbf{g}_1,g_2,g_3)$. Therefore, we have

$$G(U)\le\frac{(1+2CT(1+\frac{1}{\epsilon}))}{c_1T}(\|\mathbf{u}_T\|^2+\|\theta_T\|^2+\|S_T\|^2)$$

from which it follows that $G(U)\le C$, where C is a constant independent of U, α and β. Let us finally consider the nonlinear system (16). For a given U in $L^2(0,T;\mathbf{E}_1)$, let $(\mathbf{g},h,f)$ be the unique element such that

$$\frac{1}{2}\int_{\Gamma_N\times(0,T)}|\mathbf{g}|^2+|h|^2+|f|^2\,dx\,dt=G(U)\,. \tag{31}$$

This defines a continuous mapping $U\mapsto(\mathbf{g},h,f)$ from $L^2(0,T;\mathbf{E}_1)$ into $L^2(\Gamma_N\times(0,T))^3$. Let us denote by $(\mathbf{u}(U),\theta(U),S(U))$ the solution of (18) with control $(\mathbf{g},h,f)=(\mathbf{g}(U),h(U),f(U))$. It follows from (18) by the skew symmetry properties of the trilinear forms that

$$\left.\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|\mathbf{u}\|^2 &+ Pr\|\nabla\mathbf{u}\|^2+\tfrac{1}{\epsilon}\|\mathbf{u}\|^2_{0,\Gamma_N}\\ &= (Pr^2Gr_\theta(\theta+N_rS)\mathbf{i}_3,\mathbf{u})\\ &+ \tfrac{1}{\epsilon}(\mathbf{g},\mathbf{u})_{\Gamma_N}\,,\\ \tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 &+ \|\nabla\theta\|^2+\tfrac{1}{\epsilon}\|\theta\|^2_{0,\Gamma_N}\\ &= \tfrac{1}{\epsilon}(h,\theta)_{\Gamma_N}\,,\\ \tfrac{1}{2}\tfrac{d}{dt}\|S\|^2 &+ \tfrac{1}{Le}\|\nabla S\|^2+\tfrac{1}{\epsilon}\|S\|^2_{0,\Gamma_N}\\ &= \tfrac{\alpha_*}{Le}(\nabla\theta_m,\nabla S)+\tfrac{1}{\epsilon}(f,S)_{\Gamma_N}\,.\end{aligned}\right\} \tag{32}$$

Estimating the terms on the right hand side of the equations in (32) and arguing as in the proof of Proposition 1 we obtain

$$\begin{aligned}\sup_{t\in[0,T]}(\|\mathbf{u}\|^2 &+ \|\theta\|^2+\|S\|^2)\\ &\le \tfrac{C}{\epsilon}(\|\mathbf{g}\|^2_{L^2(0,T;L^2(\Gamma_N))}\\ &+ \|h\|^2_{L^2(0,T;L^2(\Gamma_N))}\\ &+ \|f\|^2_{L^2(0,T;L^2(\Gamma_N))})\,,\end{aligned} \tag{33}$$

for some constant C. In view of the uniform estimate (23), we have from (33) that, when U varies in $L^2(0,T;\mathbf{E}_1)$, $(\mathbf{u},\theta,S)$ remains in a bounded subset $\mathbf{S}_1\times S_2\times S_3\subseteq L^2(0,T;\mathbf{E}_1)\times L^2(0,T;E_2)\times L^2(0,T;E_3)$. Let us next prove that $\partial_t\mathbf{u}$ remains bounded in a bounded set of $L^1(0,T;\mathbf{E}_1)$ when U varies in S_1. To this end, notice that from $(18)_1$, we have

$$\begin{aligned}|(\partial_t\mathbf{u},\mathbf{e})| &\le Pr\|\nabla\mathbf{u}\|\|\nabla\mathbf{e}\|\\ &+ \alpha\|U\|_{L^3(\Omega)}\|\nabla\mathbf{u}\|\|\mathbf{e}\|_{L^6(\Omega)}\\ &+ \alpha\|U\|_{L^3(\Omega)}\|\nabla\mathbf{e}\|\|\mathbf{u}\|_{L^6(\Omega)}\\ &+ \tfrac{1}{\epsilon}\|\mathbf{u}\|_{0,\Gamma_N}\|\mathbf{e}\|_{0,\Gamma_N}+Pr^2Gr_\theta\|\theta\|\|\mathbf{e}\|\\ &+ Pr^2Gr_\theta N_r\|S\|\|\mathbf{e}\|\\ &+ \tfrac{1}{\epsilon}\|\mathbf{g}\|_{0,\Gamma_N}\|\mathbf{e}\|_{0,\Gamma_N}\,.\end{aligned}$$

Since $\mathbf{E}_1$ is finite dimensional and all norms are equivalent on finite dimensional spaces, we obtain

$$\begin{aligned}|(\partial_t\mathbf{u},\mathbf{e})| &\le C\,[Pr\|\nabla\mathbf{u}\|+\|\nabla\mathbf{u}\|\|U\|\\ &+ \tfrac{1}{\epsilon}(\|\mathbf{u}\|_{0,\Gamma_N}+\|\mathbf{g}\|_{0,\Gamma_N})\\ &+ Pr^2Gr_\theta(\|\theta\|+N_r\|S\|)]\,\|\mathbf{e}\|\end{aligned}$$

and thus

$$\begin{aligned}\|\partial_t\mathbf{u}\| &\le C\,(\|\mathbf{u}\|+\|\mathbf{u}\|\|U\|\\ &+ \tfrac{1}{\epsilon}(\|\mathbf{u}\|_{0,\Gamma_N}+\|\mathbf{g}\|_{0,\Gamma_N})\\ &+ (\|\theta\|+N_r\|S\|))\,.\end{aligned}$$

This proves that $\partial_t\mathbf{u}$ remains bounded in a bounded set of $L^1(0,T;\mathbf{E}_1)$ when U varies in S_1. Let us now define a mapping Q from S_1 to S_1 by $U\mapsto\mathbf{u}(U)$. Then the range of Q is relatively compact in S_1 by Aubin-Simon's lemma. Schauder fixed point theorem now implies that Q has a fixed point in S_1. Therefore since (18) is exactly controllable in $T>0$, we have that (17) is exactly controllable. □

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