An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.14, No.3, pp.229-248 (2024)
http://doi.org/10.11121/ijocta.1482

RESEARCH ARTICLE

Further refinements and inequalities of Fejér's type via GA-convexity

Muhammad Amer Latif [a], Hüseyin Budak [b], Artion Kashuri [c*]

[a] *Basic Sciences Unit, Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa, Saudi Arabia*
[b] *Department of Mathematics, Faculty of Science and Arts, Duzce University, Duzce 81620, Türkiye*
[c] *Department of Mathematical Engineering, Polytechnic University of Tirana, Tirana 1001, Albania*
mlatif@kfu.edu.sa, hsyn.budak@gmail.com, a.kashuri@fimif.edu.al

ARTICLE INFO

Article History:
Received 3 November 2023
Accepted 28 April 2024
Available Online 12 July 2024

Keywords:
Hermite-Hadamard inequality
Convex function
GA-convex function
Fejér inequality

AMS Classification 2010:
26D07; 26D15; 26D20

ABSTRACT

In this study, we introduce some new mappings in connection with Hermite-Hadamard and Fejér type integral inequalities which have been proved using the GA-convex functions. As a consequence, we obtain certain new inequalities of the Fejér type that provide refinements of the Hermite-Hadamard and Fejér type integral inequalities that have already been obtained.

1. Introduction

For convex functions the following double inequality has great significance in literature and is known as Hermite-Hadamard's inequality [1,2]:

Let $\tau : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$, be a convex function, then

$$\tau\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\tau(\nu)d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}. \quad (1)$$

The inequality (1) holds in reversed direction if τ is concave.
Fejér [3], established the following double inequality as a weighted generalization of (1):

$$\tau\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\int_{\varkappa_1}^{\varkappa_2} r(\nu)d\nu \leq \int_{\varkappa_1}^{\varkappa_2}\tau(\nu)r(\nu)d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}(\nu)d\nu, \quad (2)$$

where $\tau : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$ is any convex function and $r : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\nu = \frac{\varkappa_1+\varkappa_2}{2}$.

These inequalities have many extensions and generalizations, see [4]- [50]. Dragomir *et al.* [7], obtained the refinement of the first inequality in (1). Yang and Hong [42], obtained the following Hermite-Hadamard-type inequality which is a refinement of the second inequality in (1). Tseng *et al.* [35], established the Fejér-type inequalities that refined 2. Yang and Tseng [42] and

*Corresponding Author

Tseng *et al.* [35] established the Fejér-type inequalities which are weighted generalizations of results from [7] and [42]. Dragomir *et al.* [12] provided further Hermite-Hadamard-type inequality related to (1) that refine the second inequality in (1). Tseng et al. [36,37], obtained some very fascinating results related to Fejér's result (2) which are weighted generalizations of a result proven in [12]. Tseng *et al.* [38] considered the following mappings defined over an interval $[0,1]$ and discussed important results that characterize the properties of the those mappings and also proved Fejér-type inequalities that provide refinements of the Hermite-Hadamard's (1) and Fejér's inequality (2):

$$G(\alpha) := \frac{1}{2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right)\right],$$

$$Q(\alpha) := \frac{1}{2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\varkappa_2\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\varkappa_1\right)\right],$$

$$H(\alpha) := \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\tau\left(\alpha\nu + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right)d\nu,$$

$$H_r(\alpha) := \int_{\varkappa_1}^{\varkappa_2}\tau\left(\alpha\nu + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right)r(\nu)\,d\nu,$$

$$I(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\frac{\varkappa_1+\nu}{2} + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right) + \tau\left(\alpha\frac{\varkappa_2+\nu}{2} + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right)\right]r(\nu)\,d\nu,$$

$$P(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\left(\frac{1+\alpha}{2}\right)\varkappa_1 + \left(\frac{1-\alpha}{2}\right)\nu\right) + \tau\left(\left(\frac{1+\alpha}{2}\right)\varkappa_2 + \left(\frac{1-\alpha}{2}\right)\nu\right)\right]d\nu,$$

$$P_r(\alpha) := \frac{1}{2(\varkappa_2-\varkappa_1)} \times\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\left(\frac{1+\alpha}{2}\right)\varkappa_1 + \left(\frac{1-\alpha}{2}\right)\nu\right)r\left(\frac{\varkappa_1+\nu}{2}\right) + \tau\left(\left(\frac{1+\alpha}{2}\right)\varkappa_2 + \left(\frac{1-\alpha}{2}\right)\nu\right)r\left(\frac{\nu+\varkappa_2}{2}\right)\right]d\nu,$$

$$N(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\frac{\varkappa_1+\nu}{2}\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\frac{\nu+\varkappa_2}{2}\right)\right]r(\nu)\,d\nu,$$

$$L(\alpha) := \frac{1}{2(\varkappa_2-\varkappa_1)} \times\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\nu\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\nu\right)\right]d\nu,$$

$$L_r(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\nu\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\nu\right)\right]r(\nu)\,d\nu$$

and

$$S_r(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\frac{\varkappa_1+\nu}{2}\right) + \tau\left(\alpha\varkappa_1 + (1-\alpha)\frac{\nu+\varkappa_2}{2}\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\frac{\varkappa_1+\nu}{2}\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\frac{\nu+\varkappa_2}{2}\right)\right]r(\nu)\,d\nu,$$

where $\tau : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a convex function and $r : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\nu = \frac{\varkappa_1+\varkappa_2}{2}$.

Remark 1. *It should be noted that* $H = H_r = I$, $P = P_r = N$ *and* $L = L_r = S_r$ *on* $[0,1]$ *as* $r(\nu) = \frac{1}{\varkappa_2-\varkappa_1}$, $\nu \in [\varkappa_1, \varkappa_2]$.

Tseng *et al.* [38], proved new Fejér-type inequalities related to the mappings G, Q, H_r, P_r, I, N, L_r and S_r defined above. These results generalize known results obtained in connection to the Hermite-Hadamard inequality and therefore are useful in obtaining various results for means for a given convex function τ and particular weight function r.

Here we point out few important findings from Tseng *et al.* [35, 39] that were used to prove results from [38].

Lemma 1. *[35] Let* $\tau : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ *be a convex function and let* $\varkappa_1 \le \kappa_1 \le \nu_1 \le \nu_2 \le \kappa_2 \le \varkappa_2$ *with* $\nu_1 + \nu_2 = \kappa_1 + \kappa_2$. *Then*

$$\tau(\nu_1) + \tau(\nu_2) \le \tau(\kappa_1) + \tau(\kappa_2).$$

The assumptions in Lemma 1 can be weakened as in the following lemma:

Lemma 2. *[39] Let* $\tau : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ *be a convex function and let* $\varkappa_1 \le \kappa_1 \le \nu_1 \le \kappa_2 \le \varkappa_2$ *and* $\varkappa_1 \le \kappa_1 \le \nu_2 \le \kappa_2 \le \varkappa_2$ *with* $\nu_1 + \nu_2 = \kappa_1 + \kappa_2$. *Then*

$$\tau(\nu_1) + \tau(\nu_2) \le \tau(\kappa_1) + \tau(\kappa_2).$$

Lemma 3. *[39] Let* τ, G, Q *be defined as above. Then* Q *is symmetric about* $\frac{1}{2}$, Q *is decreasing on* $\left[0, \frac{1}{2}\right]$ *and increasing on* $\left[\frac{1}{2}, 1\right]$,

$$G(2\alpha) \le Q(\alpha), \quad \alpha \in \left[0, \frac{1}{4}\right],$$

$$G(2\alpha) \geq Q(\alpha), \quad \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right],$$

$$G(2(1-\alpha)) \geq Q(\alpha), \quad \alpha \in \left[\frac{1}{2}, \frac{3}{4}\right]$$

and

$$G(2(1-\alpha)) \leq Q(\alpha), \quad \alpha \in \left[\frac{3}{4}, 1\right].$$

Here we cite two important results form Tseng *et al.* [38].

Theorem 1. *[38] Let τ, r, H, P_r, L_r and S_r be defined as above. Then*

(i) *The inequality*

$$\begin{aligned} \int_{\varkappa_1}^{\varkappa_2} \tau(\nu) r(\nu) d\nu &\leq 2\left[\int_{\varkappa_1}^{\frac{3\varkappa_1+\varkappa_2}{4}} \tau(\nu) r(2\nu - \varkappa_1) d\nu + \int_{\frac{\varkappa_1+3\varkappa_2}{4}}^{\varkappa_2} \tau(\nu) r(\varkappa_2 - 2\nu) d\nu\right] \\ &\leq \int_0^1 P_r(\alpha) d\alpha \leq \frac{1}{2}\left[\int_{\varkappa_1}^{\varkappa_2} \tau(\nu) r(\nu) d\nu + \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu\right] \end{aligned} \tag{3}$$

holds.

(ii) *The inequalities*

$$\begin{aligned} L_r(\alpha) \leq P_r(\alpha) &\leq (1-\alpha) \int_{\varkappa_1}^{\varkappa_2} \tau(\nu) r(\nu) d\nu + \alpha \cdot \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \end{aligned} \tag{4}$$

and

$$\begin{aligned} 0 \leq N(\alpha) - G(\alpha) \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu - N(\alpha) \end{aligned} \tag{5}$$

hold for all $\alpha \in [0,1]$.

(iii) *If τ is differentiable on $[\varkappa_1, \varkappa_2]$, then we have the inequalities*

$$\begin{aligned} 0 &\leq \alpha \left[\frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \tau(\nu) d\nu - \tau\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right] \inf_{\nu \in [\varkappa_1, \varkappa_2]} r(\nu) \\ &\leq P_r(\alpha) - \int_{\varkappa_1}^{\varkappa_2} \tau(\nu) r(\nu) d\nu, \end{aligned} \tag{6}$$

$$\begin{aligned} 0 \leq P_r(\alpha) - \tau\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \end{aligned} \tag{7}$$

$$\begin{aligned} 0 \leq L_r(\alpha) - H_r(\alpha) \leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \end{aligned} \tag{8}$$

$$\begin{aligned} 0 \leq P_r(\alpha) - L_r(\alpha) \leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \end{aligned} \tag{9}$$

$$\begin{aligned} 0 \leq P_r(\alpha) - H_r(\alpha) \leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \end{aligned} \tag{10}$$

$$\begin{aligned} 0 \leq N(\alpha) - I(\alpha) \leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \end{aligned} \tag{11}$$

and

$$\begin{aligned} 0 \leq S_r(\alpha) - I(\alpha) \leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \end{aligned} \tag{12}$$

hold for all $\alpha \in [0,1]$.

Theorem 2. *[38] Let τ, r, G, Q, H_r, P_r and S_r be defined as above. Then*

(i) *The inequalities*

$$\begin{aligned} H_r(\alpha) \leq Q(\alpha) \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \quad \alpha \in \left[0, \frac{1}{3}\right] \end{aligned} \tag{13}$$

and

$$\tau\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\int_{\varkappa_1}^{\varkappa_2} r(\nu)\,d\nu \le Q(\alpha)\int_{\varkappa_1}^{\varkappa_2} r(\nu)\,d\nu \le P_r(\alpha), \quad \alpha\in\left[\frac{1}{3},1\right] \quad (14)$$

hold.

(ii) *The inequality*

$$0\le S_r(\alpha)-G(\alpha)\int_{\varkappa_1}^{\varkappa_2} r(\nu)\,d\nu \le \frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+Q(\alpha)\right]\times\int_{\varkappa_1}^{\varkappa_2} r(\nu)\,d\nu - S_r(\alpha) \quad (15)$$

hold for all $\alpha\in[0,1]$.

Convex functions are a fundamental concept in mathematics, and geometrically-arithmetically convex functions, or GA-convex functions, represent an exciting generalization of this concept that offers new insights and applications.

Definition 1. *[7] Suppose* $I\subseteq(0,\infty)$ *is an interval of positive real numbers. A function* $\tau: I\to\mathbb{R}$ *is considered to be GA-convex, if*

$$\tau\left(\nu^{\alpha}\kappa^{1-\alpha}\right)\le\alpha\tau(\nu)+(1-\alpha)\tau(\kappa) \quad (16)$$

for all $\nu,\kappa\in I$ *and* $\alpha\in[0,1]$. *A function* $\tau: I\to\mathbb{R}$ *is GA-concave if the inequality in (16) reversed.*

We have gathered crucial information regarding GA-convex and convex functions, which we will utilize to demonstrate our main findings.

Theorem 3. *[7] If* $[\varkappa_1,\varkappa_2]\subset(0,\infty)$ *and the function* $\mathcal{G}:[\ln\varkappa_1,\ln\varkappa_2]\to\mathbb{R}$ *is convex (concave) on* $[\ln\varkappa_1,\ln\varkappa_2]$, *then the function* $\tau:[\varkappa_1,\varkappa_2]\to\mathbb{R}, \tau(\alpha)=\mathcal{G}(\ln\alpha)$ *is GA-convex (concave) on* $[\varkappa_1,\varkappa_2]$.

Remark 2. *It is obvious from Theorem 3 that if* $\tau:[\varkappa_1,\varkappa_2]\to\mathbb{R}$ *is GA-convex on* $[\varkappa_1,\varkappa_2]\subset(0,\infty)$, *then* $\tau\circ\exp$ *is convex on* $[\ln\varkappa_1,\ln\varkappa_2]$. *It follows that* $\tau\circ\exp$ *has finite lateral derivatives on* $(\ln\varkappa_1,\ln\varkappa_2)$ *and by gradient inequality for convex functions, we have*

$$\tau\circ\exp(\nu)-\tau\circ\exp(\kappa)(\nu-\kappa)\ge\varphi(\exp\kappa)\exp(\kappa),$$

where $\varphi(\exp\kappa)\in\left[\tau'_{-}(\exp\kappa),\tau'_{+}(\exp\kappa)\right]$ *for any* $\nu,\kappa\in(\ln\varkappa_1,\ln\varkappa_2)$.

The following inequality of Hermite-Hadamard type for *GA*-convex functions holds (see [31] for an extension for GA *h*-convex functions):

Theorem 4. *[31] Let* $\tau: I\subseteq(0,\infty)\to\mathbb{R}$ *be a GA-convex function and* $\varkappa_1,\varkappa_2\in I$ *with* $\varkappa_1<\varkappa_2$. *If* $\tau\in L([\varkappa_1,\varkappa_2])$, *then the following inequalities hold:*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\le\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_2}^{\varkappa_1}\frac{\tau(\nu)}{\nu}d\nu \le\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}. \quad (17)$$

The notion of geometrically symmetric functions was introduced in [25].

Definition 2. *[25] A function* $r:[\varkappa_1,\varkappa_2]\subseteq(0,\infty)\to\mathbb{R}$ *is geometrically symmetric with respect to* $(0,\infty)$, *if*

$$r(\nu)=r\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)$$

holds for all $\nu\in[\varkappa_1,\varkappa_2]$.

Fejér type inequalities using GA-convex functions using geometrically symmetric functions were presented in Latif *et al.* [25].

Theorem 5. *[25] Let* $\tau: I\subseteq(0,\infty)\to\mathbb{R}$ *be a GA-convex function and* $\varkappa_1,\varkappa_2\in I$ *with* $\varkappa_1<\varkappa_2$. *If* $\tau\in L([\varkappa_1,\varkappa_2])$ *and* $r:[\varkappa_1,\varkappa_2]\subseteq(0,\infty)\to\mathbb{R}$ *is nonnegative, integrable and geometrically symmetric with respect to* $\sqrt{\varkappa_1\varkappa_2}$, *then*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_2}^{\varkappa_1}\frac{r(\nu)}{\nu}d\nu\le\int_{\varkappa_2}^{\varkappa_1}\frac{\tau(\nu)r(\nu)}{\nu}d\nu \le\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_2}^{\varkappa_1}\frac{r(\nu)}{\nu}d\nu. \quad (18)$$

Suppose that $\tau: I\subseteq(0,\infty)\to\mathbb{R}$ is *GA*-convex on I and $\varkappa_1,\varkappa_2\in I$, let $\mathcal{H},\mathcal{F}$,$\varkappa_{\in}$,$\mathcal{I}_r:[0,1]\to\mathbb{R}$ be defined by

$$\mathcal{H}(\alpha):=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{1}{\nu}\tau\left(\nu^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)d\nu,$$

$$\mathcal{F}(\alpha):=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}\frac{1}{\nu\kappa}\tau\left(\nu^{\alpha}\kappa^{1-\alpha}\right)d\nu d\kappa,$$

$$\mathcal{P}(\alpha):=\frac{1}{2(\ln\varkappa_2-\ln\varkappa_1)}\times\int_{\varkappa_1}^{\varkappa_2}\frac{1}{\nu}\left[\tau\left(\varkappa_2^{\frac{1+\alpha}{2}}\nu^{\frac{1-\alpha}{2}}\right)+\tau\left(\varkappa_1^{\frac{1+\alpha}{2}}\nu^{\frac{1-\alpha}{2}}\right)\right]d\nu$$

and

$$\mathcal{I}_r(\alpha):=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\left(\sqrt{\varkappa_1\nu}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\left(\sqrt{\nu\varkappa_2}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\nu,$$

where $r:[\varkappa_1,\varkappa_2]\subseteq(0,\infty)\to\mathbb{R}$ is nonnegative, integrable and geometrically symmetric with respect to $\sqrt{\varkappa_1\varkappa_2}$.

Latif *et al.* [21] obtained the following refinements for the inequalities (17):

Theorem 6. *[21] Let the function $\tau : I \subseteq (0,\infty) \to \mathbb{R}$ be as above. Then*

(i) *$\mathcal{H}$ is GA-convex on $[0,1]$.*

(ii) *We have*

$$\inf_{\alpha\in[0,1]} \mathcal{H}(\alpha) = \mathcal{H}(0) = \tau\left(\sqrt{\varkappa_1\varkappa_2}\right)$$

and

$$\sup_{\alpha\in[0,1]} \mathcal{H}(\alpha) = \mathcal{H}(1) = \frac{1}{\ln\varkappa_2 - \ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu)}{\nu} d\nu.$$

(iii) *$\mathcal{H}$ increases monotonically on $[0,1]$.*

The following theorem holds:

Theorem 7. *[21] Let $\tau : [\varkappa_1, \varkappa_2] \subseteq (0,\infty) \to \mathbb{R}$ be as above. Then*

(i) *$\mathcal{F}(\alpha + \frac{1}{2}) = \mathcal{F}(\frac{1}{2} - \alpha)$ for all α in $[0, \frac{1}{2}]$.*

(ii) *$\mathcal{F}$ is GA-convex on $[0,1]$.*

(iii) *We have*

$$\sup_{\alpha\in[0,1]} \mathcal{F}(\alpha) = \mathcal{F}(0) = \mathcal{F}(1) = \frac{1}{(\ln\varkappa_2 - \ln\varkappa_1)^2}\int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu}\tau(\nu)\, d\nu$$

and

$$\inf_{\alpha\in[0,1]} \mathcal{F}(\alpha) = \mathcal{F}\left(\frac{1}{2}\right) = \frac{1}{\ln\varkappa_2 - \ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu\kappa}\tau\left(\sqrt{\nu\kappa}\right) d\nu d\kappa.$$

(iv) *The inequality*

$$\tau\left(\sqrt{\nu\kappa}\right) \le \mathcal{F}\left(\frac{1}{2}\right)$$

is valid.

(v) *$\mathcal{F}$ decreases monotonically on $[0, \frac{1}{2}]$ and increases monotonically on $\left[\frac{1}{2}, 1\right]$.*

(vi) *The inequality $\mathcal{H}(\alpha) \le \mathcal{F}(\alpha)$ holds true for all $\alpha \in [0,1]$.*

Theorem 8. *[21] Let $\mathcal{P} : [0,1] \to \mathbb{R}$ and $\tau : [\varkappa_1, \varkappa_2] \subset (0,\infty) \to \mathbb{R}$ be defined as above. Then*

(i) *$\mathcal{P}$ is GA-convex on $(0,1]$.*

(ii) *The following hold:*

$$\inf_{\alpha\in[0,1]} \mathcal{P}(\alpha) = \mathcal{P}(0) = \frac{1}{\ln\varkappa_2 - \ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu)}{\nu} d\nu$$

and

$$\sup_{\alpha\in[0,1]} \mathcal{P}(\alpha) = \mathcal{P}(1) = \frac{\tau(\varkappa_1) + \tau(\varkappa_2)}{2}.$$

(iii) *$\mathcal{P}$ increases monotonically on $[0,1]$.*

Theorem 9. *[27] Let $\tau, r, \mathcal{I}_r$ be defined as above. Then $\mathcal{I}_r$ is GA-convex, increasing on $[0,1]$ and for all $\alpha \in [0,1]$, we have the following Fejér type inequalities:*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu \le \mathcal{I}_r(0) \le \mathcal{I}_r(\alpha)$$
$$\le \mathcal{I}_r(1) = \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\sqrt{\varkappa_1\nu}\right) + \tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu} d\nu. \tag{19}$$

Mathematical inequalities are very useful tools in establishing a number of important results in various branches of mathematical and physical sciences. Later on, mathematicians observed that the convexity plays an important role to prove novel results in theory of inequalities. Moreover, over the past three decades researchers are trying to generalize the classical convexity notion and convex functions so that they can prove new generalized and novel results in the field of mathematical inequalities that can also serve as refinements of previously proven results.

Indeed there are several generalizations of the classical convexity and convex functions in classical sense but one of them is know as the geometrically-arithmetically convexity (GA-covexity) and geometrically-arithmetically convex functions (GA-covex functions). Using the notion of GA-covexity, Noor et al. [31] and Latif et al. [25] proved results of Hermite-Hadamard and Féjer type.

The study explores a recent research study that builds upon previous work and offers fresh insights. Using their knowledge of studies conducted in [10–15, 17, 18, 34–39, 41–45], we define new mappings pertaining to two specific inequalities, namely (17) and (18). We then utilize these mappings to establish new Féjer type inequalities for GA-convex functions, employing innovative techniques and a variant of Lemma 4 for GA-convex functions to achieve results that refine (17) and (18) that are variants of inequalities given in Theorems 1 and Theorem 2. The researchers also highlight the implications of their findings and suggest future research directions, underscoring their commitment to advancing the field and making meaningful contributions.

2. Main Results

Let $\tau : [\varkappa_1, \varkappa_2] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex mapping and let $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{H}_r$, $\mathcal{I}_r$, $\mathcal{P}$, $\mathcal{P}_r$, $\mathcal{N}$, $\mathcal{S}_r : [0,1] \to \mathbb{R}$ be defined by

$$\mathcal{G}(\alpha) := \frac{1}{2}\left[\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)\right],$$

$$\mathcal{Q}(\alpha) := \frac{1}{2}\left[\tau\left(\varkappa_1^{\alpha}\varkappa_2^{1-\alpha}\right) + \tau\left(\varkappa_2\varkappa_1^{1-\alpha}\right)\right],$$

$$\mathcal{H}(\alpha) := \frac{1}{\ln \varkappa_2 - \ln \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu} \tau\left(\nu^{\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{1-\alpha}\right) d\nu,$$

$$\mathcal{H}_r(\alpha) := \int_{\varkappa_1}^{\varkappa_2} \tau\left(\nu^{\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{1-\alpha}\right) \frac{r(\nu)}{\nu} d\nu,$$

$$\mathcal{P}_r(\alpha) := \frac{1}{2} \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\frac{1+\alpha}{2}} \nu^{\frac{1-\alpha}{2}}\right) \frac{r\left(\sqrt{\varkappa_1 \nu}\right)}{\nu} + \tau\left(\varkappa_2^{\frac{1+\alpha}{2}} \nu^{\frac{1-\alpha}{2}}\right) \frac{r\left(\sqrt{\nu \varkappa_2}\right)}{\nu^2}\right] d\nu,$$

$$\mathcal{N}(\alpha) := \frac{1}{2} \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\alpha} \left(\sqrt{\varkappa_1 \nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \left(\sqrt{\nu \varkappa_2}\right)^{1-\alpha}\right)\right] \frac{r(\nu)}{\nu} d\nu,$$

$$\mathcal{L}(\alpha) := \frac{1}{2(\ln \varkappa_2 - \ln \varkappa_1)} \times \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\alpha} \nu^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \nu^{1-\alpha}\right)\right] \frac{d\nu}{\nu},$$

$$\mathcal{L}_r(\alpha) := \frac{1}{2} \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\alpha} \nu^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \nu^{1-\alpha}\right)\right] \frac{r(\nu)}{\nu} d\nu$$

and

$$\mathcal{S}_r(\alpha) := \frac{1}{2} \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\alpha} \left(\sqrt{\varkappa_1 \nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha} \left(\sqrt{\nu \varkappa_2}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \left(\sqrt{\varkappa_1 \nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \left(\sqrt{\nu \varkappa_2}\right)^{1-\alpha}\right)\right] \frac{r(\nu)}{\nu} d\nu.$$

Remark 3. *It should be noted that* $\mathcal{H} = \mathcal{H}_r = \mathcal{I}_r$, $\mathcal{P} = \mathcal{P}_r = \mathcal{N}$ *and* $\mathcal{L} = \mathcal{L}_r = \mathcal{S}_r$ *on* $[0,1]$ *when we take* $r(\nu) = \frac{1}{\ln \varkappa_2 - \ln \varkappa_1}$, $\nu \in [\varkappa_1, \varkappa_2]$.

In order to obtain the results of this section the author proved the following important lemma:

Lemma 4. *[27] Let* $\tau : [\varkappa_1, \varkappa_2] \subset (0, \infty) \to \mathbb{R}$ *be a GA-convex function and let* $\varkappa_1 \le \kappa_1 \le \nu_1 \le \nu_2 \le \kappa_2 \le \varkappa_2$ *with* $\nu_1 \nu_2 = \kappa_1 \kappa_2$. *Then*

$$\tau(\nu_1) + \tau(\nu_2) \le \tau(\kappa_1) + \tau(\kappa_2).$$

The assumptions in Lemma 4 can be weakened as in the following lemma:

Lemma 5. *Let* $\tau : [\varkappa_1, \varkappa_2] \subset (0, \infty) \to \mathbb{R}$ *be a GA-convex function and let* $\varkappa_1 \le \kappa_1 \le \nu_1 \le \kappa_2 \le \varkappa_2$ *and* $\varkappa_1 \le \kappa_1 \le \nu_2 \le \kappa_2 \le \varkappa_2$ *with* $\nu_1 \nu_2 = \kappa_1 \kappa_2$. *Then*

$$\tau(\nu_1) + \tau(\nu_2) \le \tau(\kappa_1) + \tau(\kappa_2).$$

We also need the following new lemma to prove our main results.

Lemma 6. *Let* τ, $\mathcal{G}$, $\mathcal{Q}$ *be defined as above. Then* $\mathcal{Q}$ *is symmetric about* $\frac{1}{2}$, Q *is decreasing on* $\left[0, \frac{1}{2}\right]$ *and increasing on* $\left[\frac{1}{2}, 1\right]$. *Moreover the following inequalities hold:*

$$\mathcal{G}(2\alpha) \le \mathcal{Q}(\alpha), \quad \alpha \in \left[0, \frac{1}{4}\right], \tag{20}$$

$$\mathcal{G}(2\alpha) \ge \mathcal{Q}(\alpha), \quad \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right], \tag{21}$$

$$\mathcal{G}(2(1-\alpha)) \ge \mathcal{Q}(\alpha), \quad \alpha \in \left[\frac{1}{2}, \frac{3}{4}\right] \tag{22}$$

and

$$\mathcal{G}(2(1-\alpha)) \le \mathcal{Q}(\alpha), \quad \alpha \in \left[\frac{3}{4}, 1\right]. \tag{23}$$

Proof. The GA-convexity of $\mathcal{Q}(\alpha)$ on $(0,1]$ follows from the GA-convexity of τ on $[\varkappa_1, \varkappa_2]$. It is clear that $\mathcal{Q}(\alpha)$ is symmetric about $\frac{1}{2}$. Let $0 < \alpha_1 < \alpha_2 \le \frac{1}{2} \le \alpha_3 < \alpha_4 \le 1$, then according to Lemma 4, the following inequalities hold:
The inequality

$$\tau\left(\varkappa_2^{\alpha_2} \varkappa_1^{1-\alpha_2}\right) + \tau\left(\varkappa_1^{\alpha_2} \varkappa_2^{1-\alpha_2}\right) \le \tau\left(\varkappa_2^{\alpha_1} \varkappa_1^{1-\alpha_1}\right) + \tau\left(\varkappa_1^{\alpha_1} \varkappa_2^{1-\alpha_1}\right)$$

holds for $\nu_1 = \varkappa_2^{\alpha_2} \varkappa_1^{1-\alpha_2}$, $\nu_2 = \varkappa_1^{\alpha_2} \varkappa_2^{1-\alpha_2}$, $\kappa_1 = \varkappa_2^{\alpha_1} \varkappa_1^{1-\alpha_1}$, $\kappa_2 = \varkappa_1^{\alpha_1} \varkappa_2^{1-\alpha_1}$.
The inequality

$$\tau\left(\varkappa_2^{\alpha_3} \varkappa_1^{1-\alpha_3}\right) + \tau\left(\varkappa_1^{\alpha_3} \varkappa_2^{1-\alpha_3}\right) \le \tau\left(\varkappa_2^{\alpha_4} \varkappa_1^{1-\alpha_4}\right) + \tau\left(\varkappa_1^{\alpha_4} \varkappa_2^{1-\alpha_4}\right)$$

holds for $\nu_1 = \varkappa_2^{\alpha_3} \varkappa_1^{1-\alpha_3}$, $\nu_2 = \varkappa_1^{\alpha_3} \varkappa_2^{1-\alpha_3}$, $\kappa_1 = \varkappa_2^{\alpha_4} \varkappa_1^{1-\alpha_4}$, $\kappa_2 = \varkappa_1^{\alpha_4} \varkappa_2^{1-\alpha_4}$.
Thus, $\mathcal{Q}$ is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$.
Now, we consider the following two cases:
Case 1. $\alpha \in \left[0, \frac{1}{4}\right]$
By choosing $\nu_1 = \varkappa_1^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}$, $\nu_2 = \varkappa_2^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}$, $\kappa_1 = \varkappa_2^{\alpha} \varkappa_1^{1-\alpha}$, $\kappa_2 = \varkappa_1^{\alpha} \varkappa_2^{1-\alpha}$ in Lemma 4, we get

$$\tau\left(\varkappa_1^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}\right) + \tau\left(\varkappa_2^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}\right) \le \tau\left(\varkappa_2^{\alpha} \varkappa_1^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha} \varkappa_2^{1-\alpha}\right)$$

for all $\alpha \in \left[0, \frac{1}{4}\right]$.
Case 2. $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]$
By choosing $\nu_1 = \varkappa_2^{\alpha} \varkappa_1^{1-\alpha}$, $\nu_2 = \varkappa_1^{\alpha} \varkappa_2^{1-\alpha}$, $\kappa_1 = \varkappa_1^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}$, $\kappa_2 = \varkappa_2^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}$ in Lemma 4, we get

$$\tau\left(\varkappa_2^{\alpha} \varkappa_1^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha} \varkappa_2^{1-\alpha}\right) \le \tau\left(\varkappa_1^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}\right) + \tau\left(\varkappa_2^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}\right)$$

for all $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]$.
Thus (20) and (21) are established. Using the symmetricity of $\mathcal{Q}$, (22) and (23) follow from (20) and (21), respectively. □

The author skillfully utilized Lemma 4 to obtain refined versions of Fejér type inequalities (18).

These refined inequalities not only extend the mappings related to (18), but also provide valuable insights into their properties. Overall, the author's work represents an important contribution to the field of inequalities.

Theorem 10. *[28] Let τ, $\mathcal{H}_r$, $\mathcal{P}_r$ and r be defined as above. Then*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu &= \mathcal{H}_r(0) \leq \mathcal{H}_r(\alpha)\\ \leq \mathcal{H}_r(1) = \int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu &= \mathcal{P}_r(0) \leq \mathcal{P}_r(\alpha)\\ \leq \mathcal{P}_r(1) &= \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (24)\end{aligned}$$

Theorem 11. *[27] Let τ, $\mathcal{I}_r$, $\mathcal{N}$ and r be defined as above. Then*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu &\leq \mathcal{I}_r(0) \leq \mathcal{I}_r(\alpha)\\ \leq \mathcal{I}_r(1) &= \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu\\ &= \mathcal{N}(0) \leq \mathcal{N}(\alpha) \leq \mathcal{N}(1)\\ &= \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (25)\end{aligned}$$

Corollary 1. *[27] Let τ, r be defined as above. Then, we have*

$$\begin{aligned}&\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau\left(\varkappa_1^{\frac{1}{4}}\varkappa_2^{\frac{3}{4}}\right)+\tau\left(\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{1}{2}\left[\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)+\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\right]\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (26)\end{aligned}$$

Theorem 12. *[29] Let τ, r, $\mathcal{G}$, $\mathcal{S}_r$, $\mathcal{L}_r$ be defined as above. Then, we have the following results:*

(i) *$\mathcal{L}_r$ is GA-convex on $(0,1]$.*

(ii) *The following inequalities hold for all $\alpha\in[0,1]$:*

$$\begin{aligned}\mathcal{H}_r(\alpha) &\leq \mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{L}_r(\alpha) \leq (1-\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu\\ &\quad+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (27)\end{aligned}$$

$$\mathcal{S}_r(1-\alpha) \leq \mathcal{L}_r(\alpha) \quad (28)$$

and

$$\frac{\mathcal{S}_r(\alpha)+\mathcal{S}_r(1-\alpha)}{2} \leq \mathcal{L}_r(\alpha). \quad (29)$$

(iii) *The following bound holds true:*

$$\sup_{\alpha\in[0,1]}\mathcal{L}_r(\alpha) = \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (30)$$

Theorem 13. *[29] Let τ, r, $\mathcal{G}$, $\mathcal{I}_r$, $\mathcal{S}_r$ be defined as above. Then, we have the following results:*

(i) *$\mathcal{S}_r$ is convex on $[0,1]$.*

(ii) *The following inequalities hold for all $\alpha\in[0,1]$:*

$$\begin{aligned}\mathcal{I}_r(\alpha) &\leq \mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{S}_r(\alpha)\\ &\leq (1-\alpha)\cdot\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu\\ &\quad+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (31)\end{aligned}$$

$$\mathcal{I}_r(1-\alpha) \leq \mathcal{S}_r(\alpha) \quad (32)$$

and

$$\frac{\mathcal{I}_r(\alpha)+\mathcal{I}_r(1-\alpha)}{2} \leq \mathcal{S}_r(\alpha). \quad (33)$$

(iii) *The following identity holds true:*

$$\sup_{\alpha\in[0,1]}\mathcal{S}_r(\alpha) = \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (34)$$

Now, we can prove a new variant of Theorem 1 for GA-convex functions.

Theorem 14. *Let τ, r, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{P}_r$, $\mathcal{L}_r$ and $\mathcal{S}_r$ be defined as above. Then*

(i) *The inequalities*

$$\int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu) r(\nu)}{\nu} d\nu \le 2\left[\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}} \tau(\nu) \frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu} d\nu + \int_{\varkappa_1^{\frac{1}{4}}\varkappa_2^{\frac{3}{4}}}^{\varkappa_2} \tau(\nu) \frac{r\left(\frac{\nu^2}{\varkappa_2}\right)}{\nu} d\nu\right] \le \int_0^1 \mathcal{P}_r(\alpha) d\alpha \le \frac{1}{2}\left[\int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu) r(\nu)}{\nu} d\nu + \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu\right] \quad (35)$$

hold.

(ii) *The inequalities*

$$\mathcal{L}_r(\alpha) \le \mathcal{P}_r(\alpha) \le (1-\alpha)\int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu) r(\nu)}{\nu} d\nu + \alpha \cdot \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu \le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu \quad (36)$$

and

$$0 \le \mathcal{N}(\alpha) - \mathcal{G}(\alpha) \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu \le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu - \mathcal{N}(\alpha) \quad (37)$$

hold for all $\alpha \in [0,1]$.

(iii) *If* τ *is differentiable on* $[\varkappa_1, \varkappa_2]$, *then we have the following inequalities:*

$$0 \le \alpha\left[\frac{1}{\ln \varkappa_2 - \ln \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu)}{\nu} d\nu - \tau(\sqrt{\varkappa_1 \varkappa_2})\right] \times \inf_{\nu \in [\varkappa_1, \varkappa_2]} r(\nu) \le \mathcal{P}_r(\alpha) - \int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu) r(\nu)}{\nu} d\nu, \quad (38)$$

$$0 \le \mathcal{P}_r(\alpha) - \tau(\sqrt{\varkappa_1 \varkappa_2}) \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu \le \frac{(\ln \varkappa_2 - \ln \varkappa_1)\left(\varkappa_2 \tau'(\varkappa_2) - \varkappa_1 \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu, \quad (39)$$

$$0 \le \mathcal{L}_r(\alpha) - \mathcal{H}_r(\alpha) \le \frac{(\ln \varkappa_2 - \ln \varkappa_1)\left(\varkappa_2 \tau'(\varkappa_2) - \varkappa_1 \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu, \quad (40)$$

$$0 \le \mathcal{P}_r(\alpha) - \mathcal{L}_r(\alpha) \le \frac{(\ln \varkappa_2 - \ln \varkappa_1)\left(\varkappa_2 \tau'(\varkappa_2) - \varkappa_1 \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu, \quad (41)$$

$$0 \le \mathcal{P}_r(\alpha) - \mathcal{H}_r(\alpha) \le \frac{(\ln \varkappa_2 - \ln \varkappa_1)\left(\varkappa_2 \tau'(\varkappa_2) - \varkappa_1 \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu, \quad (42)$$

$$0 \le \mathcal{N}(\alpha) - \mathcal{I}_r(\alpha) \le \frac{(\ln \varkappa_2 - \ln \varkappa_1)\left(\varkappa_2 \tau'(\varkappa_2) - \varkappa_1 \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu \quad (43)$$

and

$$0 \le \mathcal{S}_r(\alpha) - \mathcal{I}_r(\alpha) \le \frac{(\ln \varkappa_2 - \ln \varkappa_1)\left(\varkappa_2 \tau'(\varkappa_2) - \varkappa_1 \tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu \quad (44)$$

hold for all $\alpha \in [0,1]$.

Proof. (i) By using integration techniques and the assumptions on r, we get the following identities:

$$\int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu) r(\nu)}{\nu} d\nu = \int_{\varkappa_1}^{\sqrt{\varkappa_1 \varkappa_2}} \int_0^{\frac{1}{2}} \left[\tau(\nu) + \tau\left(\frac{\varkappa_1 \varkappa_2}{\nu}\right)\right] \times \frac{r(\nu)}{\nu} d\alpha d\nu, \quad (45)$$

$$2\left[\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}} \tau(\nu) \frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu} d\nu + \int_{\varkappa_1^{\frac{1}{4}}\varkappa_2^{\frac{3}{4}}}^{\varkappa_2} \tau(\nu) \frac{r\left(\frac{\nu^2}{\varkappa_2}\right)}{\nu} d\nu\right] = 2\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}} \left[\tau(\nu) + \tau\left(\frac{\varkappa_1 \varkappa_2}{\nu}\right)\right] \frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu} d\nu = 2\int_{\varkappa_1}^{\sqrt{\varkappa_1 \varkappa_2}} \int_0^{\frac{1}{2}} \left[\tau(\sqrt{\varkappa_1 \nu}) + \tau\left(\sqrt{\frac{\varkappa_1 \varkappa_2^2}{\nu}}\right)\right] \times \frac{r(\nu)}{\nu} d\alpha d\nu, \quad (46)$$

$$\int_0^1 \mathcal{P}_r(\alpha)\,d\alpha = \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^1 \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\alpha d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2} \int_0^1 \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\alpha d\nu$$
$$= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^1 \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\alpha d\nu + \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^1 \tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\alpha d\nu$$
$$= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^{\frac{1}{2}} \left[\tau\left(\varkappa_1^{1-\alpha}\nu^{\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\alpha d\nu + \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^1 \left[\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) +\tau\left(\varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\right)\right]\frac{r(\nu)}{\nu}d\alpha d\nu \tag{47}$$

and

$$\frac{1}{2}\left[\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\right] = \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^{\frac{1}{2}} \left[\tau(\varkappa_1)+\tau(\nu)\right]\frac{r(\nu)}{\nu}d\alpha d\nu + \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^{\frac{1}{2}} \left[\tau(\varkappa_2)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\frac{r(\nu)}{\nu}d\alpha d\nu. \tag{48}$$

According to Lemma 4, the following inequalities hold for all $\alpha\in\left[0,\frac{1}{2}\right]$ and $\nu\in\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$:

The inequality

$$\tau(\nu)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right) \le \tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}\right) \tag{49}$$

holds with the choices $\nu_1=\nu$, $\nu_2=\frac{\varkappa_1\varkappa_2}{\nu}$, $\kappa_1=\sqrt{\varkappa_1\nu}$ and $\kappa_2=\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}$.

The inequality

$$\tau\left(\sqrt{\varkappa_1\nu}\right) \le \frac{1}{2}\left[\tau\left(\varkappa_1^{1-\alpha}\nu^{\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\right] \tag{50}$$

holds with the choices $\nu_1=\nu_2=\sqrt{\varkappa_1\nu}$, $\kappa_1=\varkappa_1^{1-\alpha}\nu^{\alpha}$ and $\kappa_2=\varkappa_1^{\alpha}\nu^{1-\alpha}$.

The inequality

$$\tau\left(\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}\right) \le \frac{1}{2}\left[\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\right)\right] \tag{51}$$

holds with the choices $\nu_1=\nu_2=\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}$, $\kappa_1=\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$ and $\kappa_2=\varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}$.

The inequality

$$\frac{1}{2}\left[\tau\left(\varkappa_1^{1-\alpha}\nu^{\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\right] \le \frac{\tau(\varkappa_1)+\tau(\nu)}{2} \tag{52}$$

holds with the choices $\nu_1=\varkappa_1^{1-\alpha}\nu^{\alpha}$, $\nu_2=\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)$, $\kappa_1=\varkappa_1$ and $\kappa_2=\nu$.

The inequality

$$\frac{1}{2}\left[\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\right)\right] \le \frac{\tau(\varkappa_2)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)}{2} \tag{53}$$

holds with the choices $\nu_1=\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\nu_2=\varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}$, $\kappa_1=\frac{\varkappa_1\varkappa_2}{\nu}$ and $\kappa_2=\varkappa_2$.

Multiplying the inequalities (49)-(53) by $\frac{r(\nu)}{\nu}$ and integrating them over α on $\left[0,\frac{1}{2}\right]$ and over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ and using identities (45)-(48), we derive (35).

(ii) Using substitution rules for integration and the assumptions on r, we have the following identities:

$$\mathcal{P}_r(\alpha) = \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2} \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu = \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right] \times\frac{r(\nu)}{\nu}d\nu \tag{54}$$

and

$$\mathcal{L}_r(\alpha) = \frac{1}{2}\left[\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2} \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu^2}d\nu\right] + \frac{1}{2}\left[\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2} \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu\right] = \frac{1}{2}\mathcal{P}_r(\alpha) + \frac{1}{2}\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \left[\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) +\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\nu \tag{55}$$

for all $\alpha\in[0,1]$.

By choosing $\nu_1=\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\nu_2=\varkappa_2^{\alpha}\nu^{1-\alpha}$, $\kappa_1=\varkappa_1^{\alpha}\nu^{1-\alpha}$, $\kappa_2=\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$ in Lemma 5, we observe that the inequality:

$$\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\leq\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\quad(56)$$

holds for all $\alpha\in[0,1]$ and $\nu\in\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$.
Multiplying the inequality (56) by $\frac{r(\nu)}{\nu}$, integrating both sides over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ and using identities (54) and (55), we derive the first inequality of (36). The second and third inequalities of (36) can be obtained by the GA-convexity of τ and (18).
Using substitution rules for integration and the hypothesis of r, we have the following identity:

$$\begin{aligned}\mathcal{N}(\alpha)&=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}\right)^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\nu\\&=\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right]\times\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu=\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\right.\\&+\tau\left(\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right)\\&\left.+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu\quad(57)\end{aligned}$$

for all $\alpha\in[0,1]$.
By Lemma 4, the following inequalities hold for all $\alpha\in[0,1]$ and $\nu\in\left[\varkappa_1,\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}\right]$:
The inequality

$$\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right)\leq\tau(\varkappa_1)+\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)\quad(58)$$

holds for $\nu_1=\varkappa_1^{\alpha}\nu^{1-\alpha}$, $\nu_2=\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}$, $\kappa_1=\varkappa_2$ and $\kappa_2=\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$.
The inequality

$$\tau\left(\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\leq\tau(\varkappa_2)+\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)\quad(59)$$

holds for $\nu_1=\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}$, $\nu_2=\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\kappa_1=\varkappa_2$ and $\kappa_2=\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$.
Multiplying the inequalities (58)-(59) by $\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}$ and integrating them over ν on $\left[\varkappa_1,\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}\right]$ and using (57), we have

$$\mathcal{N}(\alpha)\leq\frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+\mathcal{G}(\alpha)\right]\times\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\quad(60)$$

for all $\alpha\in[0,1]$. The second inequality in (37) is a consequence of (60).
Applying Lemma 4, we observe that the inequality:

$$\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)\leq\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\quad(61)$$

holds for all $\alpha\in[0,1]$ and $\nu\in\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ when $\nu_1=\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\nu}\right)$, $\nu_2=\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$, $\kappa_1=\varkappa_1^{\alpha}\nu^{1-\alpha}$ and $\kappa_2={}^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$.

Multiplying the inequalities (61) by $\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}$ and integrating them over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ and using the first part of the identity (57), we get (37).
(iii) Integrating by parts, we have

$$\begin{aligned}&\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\frac{1}{\nu}\left(\ln\varkappa_1-\ln\nu\right)\times\left[\nu\tau'(\nu)-\frac{\varkappa_1\varkappa_2}{\nu}\tau'\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]d\nu\\&=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu-\tau\left(\sqrt{\varkappa_1\varkappa_2}\right).\quad(62)\end{aligned}$$

Using substitution rules for integration, we have the following identity:

$$\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\times\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\frac{1}{\nu}\left[\tau(\nu)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]d\nu.\quad(63)$$

Since $\tau:[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is harmonic convex on $[\varkappa_1,\varkappa_2]$, hence $g:[\ln\varkappa_1,\ln\varkappa_2]$ defined by $g(\nu):=\tau\circ\exp(\nu)$ is convex on $[\ln\varkappa_1,\ln\varkappa_2]$.
Using the convexity of g and the fact that $r(\nu)\geq 0$ on $[\ln\varkappa_1,\ln\varkappa_2]$, the inequality

$$\begin{aligned}&[g(\alpha\ln\varkappa_1+(1-\alpha)\nu)-g(\nu)]\,r(\ln\nu)\\&+[g(\alpha\ln\varkappa_2+(1-\alpha)(\ln\varkappa_1+\ln\varkappa_2-\nu))\\&-g(\ln\varkappa_1+\ln\varkappa_2-\nu)]\,r(\ln\nu)\\&\geq\alpha(\ln\varkappa_1-\nu)\,g'(\nu)\,r(\ln\nu)\\&+\alpha(\nu-\ln\varkappa_1)\,g'(\ln\varkappa_1+\ln\varkappa_2-\nu)\,r(\ln\nu)\\&=\alpha(\nu-\ln\varkappa_1)\\&\times\left[g'(\ln\varkappa_1+\ln\varkappa_2-\nu)-g'(\nu)\right]r(\ln\nu)\end{aligned}\tag{64}$$

holds for all $\alpha\in[0,1]$ and $\nu\in\left[\ln\varkappa_1,\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right]$.

The inequality (64) can be re-written as

$$\begin{aligned}&\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)-\tau(\nu)\right]\frac{r(\nu)}{\nu}\\&+\left[\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)-\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\frac{r(\nu)}{\nu}\\&\geq\nu\alpha(\ln\nu-\ln\varkappa_1)\,\tau'(\nu)\frac{r(\nu)}{\nu}\\&-\alpha(\ln\nu-\ln\varkappa_1)\frac{\varkappa_1\varkappa_2}{\nu}\tau'\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\frac{r(\nu)}{\nu}\\&=\alpha(\ln\nu-\ln\varkappa_1)\\&\times\left[\nu\tau'(\nu)-\frac{\varkappa_1\varkappa_2}{\nu}\tau'\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\frac{r(\nu)}{\nu}\\&\geq\alpha(\ln\nu-\ln\varkappa_1)\\&\times\left[\nu\tau'(\nu)-\frac{\varkappa_1\varkappa_2}{\nu}\tau'\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\\&\times\frac{1}{\nu}\inf_{\nu\in[\varkappa_1,\varkappa_2]}r(\nu)\end{aligned}\tag{65}$$

for all $\alpha\in[0,1]$ and $\nu\in\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$.
Integrating the above inequality over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$, multiplying both sides by $\frac{1}{\ln\varkappa_2-\ln\varkappa_1}$ and using (17), (54), (63) and (65), we derive (38). We also observe that

$$\begin{aligned}&\frac{g(\ln\varkappa_1)-g\left(\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)}{2}\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&\leq\frac{1}{2}\left(\ln\varkappa_1-\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)\\&\times g'(\ln\varkappa_1)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&=\left(\frac{\ln\varkappa_1-\ln\varkappa_2}{4}\right)g'(\ln\varkappa_1)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\end{aligned}\tag{66}$$

and

$$\begin{aligned}&\frac{g(\ln\varkappa_2)-g\left(\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)}{2}\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&\leq\frac{1}{2}\left(\ln\varkappa_2-\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)\\&\times g'(\ln\varkappa_2)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&=\left(\frac{\ln\varkappa_2-\ln\varkappa_1}{4}\right)g'(\ln\varkappa_2)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu.\end{aligned}\tag{67}$$

Adding (66) and (67), we get

$$\begin{aligned}&\frac{g(\ln\varkappa_1)+g(\ln\varkappa_2)}{2}\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&-g\left(\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(g'(\ln\varkappa_2)-g'(\ln\varkappa_1)\right)}{4}\\&\times\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu.\end{aligned}\tag{68}$$

The inequality (68) is equivalent to

$$\begin{aligned}&\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\&-\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\&\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4}\\&\times\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu.\end{aligned}\tag{69}$$

Finally, inequalities (39)-(44) follow from inequalities (24), (25), (27), (31), (36) and (69). □

Corollary 2. *If* $r(\nu)=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}$, $\nu\in[\varkappa_1,\varkappa_2]$, *then Hermite-Hadamard-type inequalities, that are obvious consequences of Theorem 14, are given as follows:*

(i) *The inequalities*

$$\begin{aligned}&\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\\&\leq\frac{2}{\ln\varkappa_2-\ln\varkappa_1}\end{aligned}$$

$$\times\left[\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}}\frac{\tau(\nu)}{\nu}d\nu+\int_{\varkappa_1^{\frac{1}{4}}\varkappa_2^{\frac{3}{4}}}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\right]$$

$$\leq\int_0^1\mathcal{P}(\alpha)\,d\alpha$$

$$\leq\frac{1}{2}\left[\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu+\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\right]\quad(70)$$

hold.

(ii) *The inequalities*

$$\mathcal{L}(\alpha)\leq\mathcal{P}(\alpha)\leq\frac{1-\alpha}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\quad(71)$$

and

$$0\leq\mathcal{P}(\alpha)-\mathcal{G}(\alpha)\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}-\mathcal{P}(\alpha)\quad(72)$$

hold for all $\alpha\in[0,1]$.

(iii) *If* τ *is differentiable on* $[\varkappa_1,\varkappa_2]$, *then we have the inequalities:*

$$0\leq\alpha\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\left[\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\times\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu-\tau(\sqrt{\varkappa_1\varkappa_2})\right]\leq\mathcal{P}(\alpha)-\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu,\quad(73)$$

$$0\leq\mathcal{P}(\alpha)-\tau(\sqrt{\varkappa_1\varkappa_2})\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4},\quad(74)$$

$$0\leq\mathcal{L}(\alpha)-\mathcal{H}(\alpha)\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4},\quad(75)$$

$$0\leq\mathcal{P}(\alpha)-\mathcal{L}(\alpha)\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4}\quad(76)$$

and

$$0\leq\mathcal{P}(\alpha)-\mathcal{H}(\alpha)\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4},\quad(77)$$

hold for all $\alpha\in[0,1]$.

Remark 4. *The inequality (35) gives a new refinement of the Fejér's inequality (18).*

Remark 5. *The inequality (36) refines the Fejér-type inequality (27).*

In the next theorem, we point out some inequalities for the functions $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{S}_r$ considered above.

Theorem 15. *Let* τ, r, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{S}_r$ *be defined as above. Then the following Fejér type inequalities hold true:*

(i) *The inequalities*

$$\mathcal{H}_r(\alpha)\leq\mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu,\quad(78)$$

hold for $\alpha\in\left[0,\frac{1}{3}\right]$ *and*

$$\tau(\sqrt{\varkappa_1\varkappa_2})\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\mathcal{P}_r(\alpha),\quad(79)$$

hold for $\alpha\in\left[\frac{1}{3},1\right]$.

(ii) *The inequalities*

$$0\leq\mathcal{S}_r(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+\mathcal{Q}(\alpha)\right]\times\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu+\mathcal{S}_r(\alpha),\quad(80)$$

hold for all $\alpha\in[0,1]$.

Proof. (i) Here we consider the following two cases:

Case 1. $\alpha\in\left[0,\frac{1}{3}\right]$.

Using substitution rules for integration and the hypothesis of r, we have the following identity:

$$\mathcal{H}_r(\alpha)=\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\left[\tau\left(\nu^{\alpha}(\sqrt{\varkappa_1\varkappa_2})^{1-\alpha}\right)+\tau\left(\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}(\sqrt{\varkappa_1\varkappa_2})^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\nu.\quad(81)$$

We observe that the following inequality is a result of application of Lemma 4:

The inequality

$$\tau\left(\nu^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \leq \tau\left(\varkappa_1^{1-\alpha}\varkappa_2^{\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\varkappa_2^{1-\alpha}\right) \quad (82)$$

holds for $\nu_1 = \nu^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$, $\nu_2 = \left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$, $\kappa_1 = \varkappa_1^{1-\alpha}\varkappa_2^{\alpha}$, $\kappa_2 = \varkappa_1^{\alpha}\varkappa_2^{1-\alpha}$ in Lemma 4, where $\alpha \in \left[0, \frac{1}{3}\right]$ and $\nu \in \left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$.

Multiplying the inequality (82) by $\frac{r(\nu)}{\nu}$, integrating both sides over ν on $\left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$ and using identity (81), we derive the first inequality of (78). From Lemma 6, we get that

$$\sup_{\alpha\in\left[0,\frac{1}{2}\right]} \mathcal{Q}(\alpha) = \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}.$$

Thus the second inequality in (78) is established.

Case 2. $\alpha \in \left[\frac{1}{3}, 1\right]$.

By choosing $\nu_1 = \varkappa_1^{\alpha}\varkappa_2^{1-\alpha}$, $\nu_2 = \varkappa_1^{1-\alpha}\varkappa_2^{\alpha}$, $\kappa_1 = \varkappa_1^{\alpha}\nu^{1-\alpha}$, $\kappa_2 = \varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$ in Lemma 6, where $\alpha \in \left[\frac{1}{3}, 1\right]$ and $\nu \in \left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$, we get

$$\tau\left(\varkappa_1^{\alpha}\varkappa_2^{1-\alpha}\right)+\tau\left(\varkappa_1^{1-\alpha}\varkappa_2^{\alpha}\right) \leq \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right). \quad (83)$$

Multiplying the inequality (83) by $\frac{r(\nu)}{\nu}$, integrating both sides over ν on $\left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$ and using identity (54), we derive the second inequality of (79). From Lemma 6, we get that

$$\inf_{\alpha\in\left[\frac{1}{2},1\right]} \mathcal{Q}(\alpha) = \tau\left(\sqrt{\varkappa_1\varkappa_2}\right).$$

Thus the first inequality in (79) is also achieved.

(ii) Using substitution rules for integration and the hypothesis of r, we have the following identity:

$$\begin{aligned}2\mathcal{S}_r &= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\right] \times \frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_2}\right)}{\nu}d\nu \\
&= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu \\
&= \int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right)\right. \\
&\quad + \tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right) \\
&\quad \left. +\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu. \quad (84)\end{aligned}$$

By using Lemma 4, we observe that the following inequality holds for all $\alpha \in [0,1]$ and $\nu \in \left[\varkappa_1, \varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}\right]$:

The inequality

$$\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right) \leq \tau(\varkappa_1)+\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \quad (85)$$

holds for $\nu_1 = \varkappa_1^{\alpha}\nu^{1-\alpha}$, $\nu_2 = \varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}$, $\kappa_1 = \varkappa_1$ and $\kappa_2 = \varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$.

The inequality

$$\tau\left(\varkappa_1^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) \leq \tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\varkappa_2^{1-\alpha}\right) \quad (86)$$

holds for $\nu_1 =, \varkappa_1^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}$, $\nu_2 = \varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\kappa_1 = \varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$ and $\kappa_2 = \varkappa_1^{\alpha}\varkappa_2^{1-\alpha}$.

The inequality

$$\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right)$$
$$\leq\tau\left(\varkappa_2^{\alpha}\varkappa_1^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \quad (87)$$

holds for $\nu_1=\varkappa_2^{\alpha}\nu^{1-\alpha}$, $\nu_2=\varkappa_2^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}$, $\kappa_1=\varkappa_2^{\alpha}\varkappa_1^{1-\alpha}$ and $\kappa_2=\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$.
The inequality

$$\tau\left(\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)$$
$$\leq\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2\right) \quad (88)$$

holds for $\nu_1=\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}$, $\nu_2=\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\kappa_1=\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$ and $\kappa_2=\varkappa_2$.
Multiplying the inequalities (85)-(88) by $\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}$ and integrating them over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ and using identity (84), we get

$$2\mathcal{S}_r(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+\mathcal{Q}(\alpha)\right]$$
$$\times\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (89)$$

for all $\alpha\in[0,1]$. Using (31) and (89), we derive (80). □

Corollary 3. *Let* $r(\nu)=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}$, $\nu\in[\varkappa_1,\varkappa_2]$ *in Theorem 15. Then* $\mathcal{I}_r(\alpha)=\mathcal{H}(\alpha)$, $\alpha\in[0,1]$ *and therefore we observe that:*

(i) *The inequalities*

$$\mathcal{H}(\alpha)\leq\mathcal{Q}(\alpha)\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}, \quad (90)$$

hold for $\alpha\in\left[0,\frac{1}{3}\right]$ *and*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\leq\mathcal{Q}(\alpha)\leq\mathcal{P}(\alpha), \quad (91)$$

hold for $\alpha\in\left[\frac{1}{3},1\right]$.

(ii) *The inequalities*

$$0\leq\mathcal{L}(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+\mathcal{Q}(\alpha)\right]$$
$$+\mathcal{L}(\alpha), \quad (92)$$

hold for all $\alpha\in[0,1]$.

The following Fejér-type inequalities can be deduced from Theorems 5, 10, 12, 13, 14, 15, Corollary 1 and Lemma 6 and we omit their proofs.

Theorem 16. *Let* τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{I}_r$, $\mathcal{L}_r$, $\mathcal{S}_r$ *be defined as above. Then, the following inequalities hold for all* $\alpha\in[0,1]$:

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{H}_r(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{S}_r(\alpha)\leq(1-\alpha)$$
$$\times\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu$$
$$+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (93)$$

and

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{I}_r(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{L}_r(\alpha)\leq\mathcal{P}_r(\alpha)$$
$$\leq(1-\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu$$
$$+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (94)$$

Theorem 17. *Let* τ, r, $\mathcal{H}_r$, $\mathcal{G}$, $\mathcal{I}_r$, $\mathcal{Q}$ *be defined as above. Then, the following inequalities hold for all* $\alpha\in\left[0,\frac{1}{4}\right]$:

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\mathcal{H}_r(\alpha)$$
$$\leq\mathcal{H}_r(2\alpha)\leq\mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (95)$$

and

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu &\leq \mathcal{I}_r(\alpha)\\ &\leq \mathcal{I}_r(2\alpha)\leq \mathcal{I}_r(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (96)\end{aligned}$$

Theorem 18. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha\in\left[\frac{1}{4},\frac{1}{3}\right]$:*

$$\begin{aligned}&\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{H}_r(\alpha)\leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu^2}d\nu\\ &\leq \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq \mathcal{L}_r(2\alpha)\\ &\leq \mathcal{P}_r(2\alpha)\leq (1-2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)\,r(\nu)}{\nu}d\nu\\ &+2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (97)\end{aligned}$$

and

$$\begin{aligned}&\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{H}_r(\alpha)\leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq \mathcal{S}_r(2\alpha)\leq (1-2\alpha)\\ &\times\int_{\varkappa_1}^{\varkappa_2}\frac{1}{2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu\\ &+2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (98)\end{aligned}$$

Theorem 19. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha\in\left[\frac{1}{3},\frac{1}{2}\right]$:*

$$\begin{aligned}&\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq \mathcal{L}_r(2\alpha)\\ &\leq \mathcal{P}_r(2\alpha)\leq (1-2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)\,r(\nu)}{\nu}d\nu\\ &+2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (99)\end{aligned}$$

$$\begin{aligned}&\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{S}_r(2\alpha)\leq (1-2\alpha)\\ &\times\int_{\varkappa_1}^{\varkappa_2}\frac{1}{2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu\\ &+2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (100)\end{aligned}$$

and

$$\begin{aligned}&\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{P}_r(\alpha)\leq \mathcal{P}_r(2\alpha)\\ &\leq (1-2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)\,r(\nu)}{\nu}d\nu\\ &+2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (101)\end{aligned}$$

Theorem 20. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha\in\left[\frac{1}{2},\frac{2}{3}\right]$:*

$$\begin{aligned}&\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq \mathcal{G}(2(1-\alpha))\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\end{aligned}$$

$$\leq \mathcal{L}_r\left(2\left(1-\alpha\right)\right) \leq \mathcal{P}_r\left(2\left(1-\alpha\right)\right) \leq \left(2\alpha-1\right)\int_{\varkappa_1}^{\varkappa_2} \frac{\tau\left(\nu\right)r\left(\nu\right)}{\nu}d\nu + 2\left(1-\alpha\right)\cdot\frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \quad (102)$$

and

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{Q}\left(\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{G}\left(2\left(1-\alpha\right)\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{S}_r\left(2\left(1-\alpha\right)\right) \leq \left(2\alpha-1\right) \times \int_{\varkappa_1}^{\varkappa_2}\frac{1}{2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r\left(\nu\right)}{\nu}d\nu + 2\left(1-\alpha\right)\cdot\frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu. \quad (103)$$

Theorem 21. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha \in \left[\frac{2}{3}, \frac{3}{4}\right]$:*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{Q}\left(\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{G}\left(2\left(1-\alpha\right)\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{G}\left(\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{L}_r\left(\alpha\right) \leq \mathcal{P}_r\left(\alpha\right) \leq \left(1-\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau\left(\nu\right)r\left(\nu\right)}{\nu}d\nu + \alpha\cdot\frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \quad (104)$$

and

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{Q}\left(\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{G}\left(2\left(1-\alpha\right)\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{G}\left(\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{S}_r\left(\alpha\right) \leq \left(1-\alpha\right) \times \int_{\varkappa_1}^{\varkappa_2}\frac{1}{2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r\left(\nu\right)}{\nu}d\nu \leq \alpha\cdot\frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu. \quad (105)$$

Theorem 22. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha \in \left[\frac{3}{4}, 1\right]$:*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{H}_r\left(2\left(1-\alpha\right)\right) \leq \mathcal{G}\left(2\left(1-\alpha\right)\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{Q}\left(\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{P}_r\left(\alpha\right) \leq \left(1-\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau\left(\nu\right)r\left(\nu\right)}{\nu}d\nu + \alpha\cdot\frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \quad (106)$$

and

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{I}_r\left(2\left(1-\alpha\right)\right) \leq \mathcal{G}\left(2\left(1-\alpha\right)\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \mathcal{Q}\left(\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu^2}d\nu \leq \mathcal{P}_r\left(\alpha\right) \leq \left(1-\alpha\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau\left(\nu\right)r\left(\nu\right)}{\nu}d\nu + \alpha\cdot\frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu \leq \frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r\left(\nu\right)}{\nu}d\nu. \quad (107)$$

Corollary 4. *Let τ, $\mathcal{Q}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{P}$, $\mathcal{L}$ be defined as above and $r\left(\nu\right) = \frac{1}{\ln\varkappa_2 - \ln\varkappa_1}$, then we have:*

(i) *The inequalities*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) \leq \mathcal{H}\left(\alpha\right) \leq \mathcal{H}\left(2\alpha\right) \leq \mathcal{G}\left(2\alpha\right) \leq \mathcal{Q}\left(\alpha\right) \leq \frac{\tau\left(\varkappa_1\right)+\tau\left(\varkappa_2\right)}{2} \quad (108)$$

hold for all $\alpha \in \left[0, \frac{1}{4}\right]$.

(ii) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{H}(\alpha) \le \mathcal{Q}(\alpha)\\ &\le \mathcal{G}(2\alpha) \le \mathcal{L}(2\alpha) \le \mathcal{P}(2\alpha)\\ &\le \left(\frac{1-2\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\\ &\quad + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (109)\end{aligned}$$

hold for all $\alpha \in \left[\frac{1}{4},\frac{1}{3}\right]$.

(iii) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{Q}(\alpha) \le \mathcal{G}(2\alpha)\\ &\le \mathcal{L}(2\alpha) \le \mathcal{P}(2\alpha)\\ &\le \left(\frac{1-2\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\\ &\quad + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (110)\end{aligned}$$

and

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{Q}(\alpha) \le \mathcal{P}(\alpha)\\ &\le \mathcal{P}(2\alpha) \le \left(\frac{1-2\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\\ &\quad + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (111)\end{aligned}$$

hold for all $\alpha \in \left[\frac{1}{3},\frac{1}{2}\right]$.

(iv) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{Q}(\alpha) \le \mathcal{G}(2(1-\alpha))\\ &\le \mathcal{L}(2(1-\alpha)) \le \mathcal{P}(2(1-\alpha))\\ &\le \left(\frac{2\alpha-1}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\\ &\quad + 2(1-\alpha)\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (112)\end{aligned}$$

hold for all $\alpha \in \left[\frac{1}{2},\frac{2}{3}\right]$.

(v) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{Q}(\alpha) \le \mathcal{G}(2(1-\alpha))\\ &\le \mathcal{G}(\alpha) \le \mathcal{L}(\alpha) \le \mathcal{P}(\alpha)\\ &\le \left(\frac{1-\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\\ &\quad + \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (113)\end{aligned}$$

hold for all $\alpha \in \left[\frac{2}{3},\frac{3}{4}\right]$.

(vi) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{H}(2(1-\alpha))\\ &\le \mathcal{G}(2(1-\alpha)) \le \mathcal{Q}(\alpha) \le \mathcal{P}(\alpha)\\ &\le \left(\frac{1-\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\\ &\quad + \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (114)\end{aligned}$$

hold for all $\alpha \in \left[\frac{3}{4},1\right]$.

3. Conclusions

Overall, this paper aimed to introduce some new mappings in connection with Hermite-Hadamard and Fejér type integral inequalities which have been proved using the GA-convex functions. As a consequence, we obtained certain new inequalities of the Fejér type that provided refinements of the Hermite-Hadamard and Fejér type integral inequalities that have already been obtained. We believe that these new techniques will be important tools for interested researcher for investigating various variational problems for different types of convexities. We hope that this research can motivate the researchers to demonstrate new results for functions of two or more variables by considering the GA-convexity and coordinated GA-convex functions on a rectangle from a plane.

References

[1] Hermite, C. (1893). Sur deux limites d'une intégrale dé finie. *Mathesis*, 3, 82.

[2] Hadamard, J. (1893). Étude sur les propriétés des fonctions entiéres en particulier d'une function considéré par Riemann. *ournal de Mathématiques Pures et Appliquées*, 9, 171–215.

[3] Fejér, L. (1906). Über die Fourierreihen, II. *Math. Naturwiss. Anz Ungar. Akad. Wiss.*, 24, 369–390.

[4] Ardic, M. A., Akdemir, A. O. & Set, E. (2016). New Ostrowski like inequalities for *GG*-convex and *GA*-convex functions. *Mathematical Inequalities & Applications*, 19(4), 1159–1168. https://doi.org/10.7153/mia-19-85

[5] Ardic, M. A., Akdemir, A. O. & Yildiz, K. (2018). On some new inequalities via *GG*-convexity and *GA*-convexity. *Filomat*, 32(16), 5707–5717. https://doi.org/10.2298/FIL1816707A

[6] Dragomir, S. S., Latif, M. A. & Momoniat, E. (2019). Fejér type integral inequalities related with geometrically-arithmetically convex functions with applications. *Acta et Commentationes Universitatis Tartuensis de Mathematica*, 23(1),

51–64. https://doi.org/10.12697/ACUTM.2019.23.05

[7] Dragomir, S. S. (2018). Some new inequalities of Hermite-Hadamard type for *GA*-convex functions. *Annales Universitatis Mariae Curie-Sklodowska Lublin-Polonia*, 72(1), 55–68. https://doi.org/10.17951/a.2018.72.1.55-68

[8] Dragomir, S. S. (2018). Inequalities of Hermite-Hadamard type for *GA*-convex functions. *Annales Mathematicae Silesianae*, 32, 145–168. https://doi.org/10.2478/amsil-2018-0001

[9] Dragomir, S. S. (2018). Inequalities of Jensen type for GA-convex functions. *Nonlinear Functional Analysis and Applications*, 23(2), 275-304.

[10] Dragomir, S. S., Cho, Y. J. & Kim, S. S. (2000). Inequalities of Hadamard's type for Lipschitzian mappings and their applications. *Journal of Mathematical Analysis and Applications*, 245, 489–501. https://doi.org/10.1006/jmaa.2000.6769

[11] Dragomir, S. S., Milosevic, D. S. & Sandor, J. (1993). On some refinements of Hadamard's inequalities and applications. *Publikacije Elektrotehničkog Fakulteta. Serija Matematika*, 4, 3–10.

[12] Dragomir, S. S. (1992). On Hadamard's inequality for convex functions. *Mathematica Balkanica*, 6, 215–222.

[13] Kashuri, A. & Liko, R. (2019). Some new Hermite-Hadamard type inequalities and their applications. *Studia Scientiarum Mathematicarum Hungarica*, 56(1), 103–142. https://doi.org/10.1556/012.2019.56.1.1418

[14] Kashuri, A., Sahoo, S. K., Mohammed, P. O., Sarairah, E. A. & Hamed, Y. S. (2023). Some new Hermite-Hadamard type inequalities pertaining to fractional integrals with an exponential kernel for subadditive functions. *Symmetry*, 15, 748. https://doi.org/10.3390/sym15030748

[15] Dragomir, S. S. (2000). On Hadamard's inequality for the convex mappings defined on a ball in the space and applications. *Mathematical Inequalities & Applications*, 3(2), 177–187. https://doi.org/10.7153/mia-03-21

[16] Dragomir, S. S. (1996). On some integral inequalities for convex functions. *Zb.-Rad. (Kragujevac)*, 18, 21–25.

[17] Dragomir, S. S. & Agarwal, R. P. (1998). Two new mappings associated with Hadamard's inequalities for convex functions. *Applied Mathematics Letters*, 11(3), 33–38. https://doi.org/10.1016/S0893-9659(98)00030-5

[18] Dragomir, S. S. (1992). Two mappings in connection to Hadamard's inequalities. *Journal of Mathematical Analysis and Applications*, 167, 49–56. https://doi.org/10.1016/0022-247X(92)90233-4

[19] Kunt, M. & İşcan, İ. (2018). Fractional Hermite-Hadamard-Fejér type inequalities for *GA*-convex functions. *Turkish Journal of Inequalities*, 2, 1–20.

[20] İşcan, İ. (2014). Hermite-Hadamard type inequalities for *GA*-*s*-convex functions. *Le Matematiche*, 19, 129–146.

[21] Latif, M. A., Kalsoom, H., Khan, Z. A., & Al-moneef, A. A. (2022). Refinement mappings related to Hermite-Hadamard type inequalities for *GA*-convex function. *Mathematics*, 10, 1398. https://doi.org/10.3390/math10091398

[22] Latif, M. A. (2014). New Hermite-Hadamard type integral inequalities for *GA*-convex functions with applications. *Analysis*, 34, 379–389. https://doi.org/10.1515/anly-2012-1235

[23] Latif, M. A., Dragomir, S. S. & Momoniat, E. (2017). Some estimates on the Hermite-Hadamard inequality through geometrically quasi-convex functions. *Miscolc Mathematical Notes*, 18(2), 933–946. https://doi.org/10.18514/MMN.2017.1819

[24] Latif, M. A. (2015). Hermite-Hadamard type inequalities for *GA*-convex functions on the coordinates with applications. *Proceedings of the Pakistan Academy of Sciences*, 52(4), 367–379.

[25] Latif, M. A., Dragomir, S. S. & Momoniat, E. (2018). Some Fejér type integral inequalities for geometrically-arithmetically-convex functions with applications. *Filomat*, 32(6), 2193–2206. https://doi.org/10.2298/FIL1806193L

[26] Latif, M. A. (2022). Weighted Hermite-Hadamard type inequalities for differentiable *GA*-convex and geometrically quasi-convex mappings. *Rocky Mountain Journal of Mathematics*, 51(6),1899–1908. https://doi.org/10.1216/rmj.2021.51.1899

[27] Latif, M. A. Fejér type inequalities for *GA*-convex functions and related results. (Submitted)

[28] Latif, M. A. Fejér type inequalities and *GA*-convex functions. (Submitted)

[29] Latif, M. A. Some companions of Fejér type inequalities using *GA*-convex functions. (Submitted)

[30] Niculescu, C. P. (2000). Convexity according to the geometric mean. *Mathematical Inequalities and Applications*, 3, 155–167. https://doi.org/10.7153/mia-03-19

[31] Noor, M. A., Noor, K. I. & Awan, M. U. (2014). Some inequalities for geometrically-arithmetically *h*-convex functions. *Creative Mathematics and Informatics*, 23(1), 91–98. https://doi.org/10.37193/CMI.2014.01.14

[32] Obeidat, S. & Latif, M. A. (2018). Weighted version of Hermite-Hadamard type inequalities for geometrically quasi-convex functions and their applications. *Journal of Inequalities and Applications*, 2018, Article 307. https://doi.org/10.1186/s13660-018-1904-7

[33] Qi, F. & Xi, B. Y. (2014). Some Hermite-Hadamard type inequalities for geometrically quasi-convex functions. *Indian Academy of Sciences Proceedings - Mathematical Sciences*, 124(3), 333–342. https://doi.org/10.1007/s12044-014-0182-7

[34] Tseng, K. L., Hwang, S. R. & Dragomir, S. S. (2007). On some new inequalities of Hermite-Hadamard- Fejér type involving convex functions. *Demonstratio Mathematica*, 40(1), 51–64. https://doi.org/10.1515/dema-2007-0108

[35] Tseng, K. L., Hwang, S. R. & Dragomir, S. S. (2010). Fejér-type inequalities (I). *Journal of Inequalities and Applications*, 2010, Article 531976. https://doi.org/10.1155/2010/531976

[36] Tseng, K. L., Hwang, S. R. & Dragomir, S. S. (2015). Some companions of Fejér's inequality for convex functions. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 109, 645–656. https://doi.org/10.1007/s13398-014-0206-2

[37] Tseng, K. L., Hwang, S. R. & Dragomir, S. S. (2017). Fejér-type Inequalities (II). *Math. Slovaca*, 67(1), 109–120. https://doi.org/10.1515/ms-2016-0252

[38] Tseng, K. L., Hwang, S. R. & Dragomir, S. S. (2011). On some weighted integral inequalities for convex functions related Fejér result. *Filomat*, 25(1), 195–218. https://doi.org/10.2298/FIL1101195T

[39] Tseng, K. L., Yang, G. S. & Hsu, K. C. (2009). On some inequalities for Hadamard's type and applications. *Taiwanese Journal of Mathematics*, 13(6B), 1929–1948. https://doi.org/10.11650/twjm/1500405649

[40] Xiang, R. (2015). Refinements of Hermite-Hadamard type inequalities for convex functions via fractional integrals. *Journal of Applied Mathematics & Informatics*, 33(1-2), 119–125. https://doi.org/10.14317/jami.2015.119

[41] Yang, G. S. & Hong, M. C. (1997). A note on Hadamard's inequality. *Tamkang Journal of Mathematics*, 28(1), 33–37. https://doi.org/10.5556/j.tkjm.28.1997.4331

[42] Yang, G. S. & Tseng, K. L. (1999). On certain integral inequalities related to Hermite-Hadamard inequalities. *Journal of Mathematical Analysis and Applications*, 239, 180–187. https://doi.org/10.1006/jmaa.1999.6506

[43] Yang, G. S. & Tseng, K. L. (2001). Inequalities of Hadamard's type for Lipschitzian mappings. *Journal of Mathematical Analysis and Applications*, 260, 230–238. https://doi.org/10.1006/jmaa.2000.7460

[44] Yang, G. S. & Tseng, K. L. (2002). On certain multiple integral inequalities related to Hermite-Hadamard inequalities. *Utilitas Mathematica*, 62, 131–142.

[45] Yang, G. S. & Tseng, K. L. (2003). Inequalities of Hermite-Hadamard-Fejér type for convex functions and Lipschitzian functions. *Taiwanese Journal of Mathematics*, 7(3), 433–440.

[46] Zhang, X. M., Chu, Y. M. & Zhang, X. H. (2010). The Hermite-Hadamard type inequality of GA-convex functions and its application. *Journal of Inequalities and Applications*, 2010, Article 507560. https://doi.org/10.1155/2010/507560

[47] Kashuri, A. & Liko, R. (2020). Fractional trapezium type inequalities for twice differentiable preinvex functions and their applications. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 10(2), 226-236. https://doi.org/10.11121/ijocta.01.2020.00795

[48] Kadakal, M. (2020). Some Hermite-Hadamard type inequalities for(P, m)-function and quasi m-convex functions. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 10(1), 78–84. https://doi.org/10.11121/ijocta.01.2020.00787

[49] Okur, N., Íscan, Í. & Dizdar, E. Y. (2019). Hermite-Hadamard type inequalities for p-convex stochastic processes. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 9(2), 148–153. https://doi.org/10.11121/ijocta.01.2019.00602

[50] Kadakal, H. (2019). Some integral inequalities for multiplicatively geometrically P-functions. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 9(2), 216–222. https://doi.org/10.11121/ijocta.01.2019.00738

Muhammad Amer Latif *is an assistant professor at Department of Basic Unit, Preparatory Year, King Faisal University. He received his Ph.D. in Mathematics from the University of Witwatersrand, Johannesburg, South Africa. His research interest is in Mathematical Inequalities, Special Functions, Fractional Calculus, Quantum Calculus, Numerical Analysis and Applied Mathematics. He has more than 150 published papers in high reputation journals.*

https://orcid.org/0000-0003-2349-3445

Hüseyin Budak *is an associate professor at Department of Mathematics, Faculty of Science and Arts, Duzce University. He received his Ph.D. in Mathematics from the Duzce University, Türkiye. His research interest is in Mathematical Inequalities, Special Functions, Fractional Calculus, Quantum Calculus, Qualitative Analysis, Numerical Analysis and Applied Mathematics. He has more than 400 published papers in high reputation journals.*

https://orcid.org/0000-0001-8843-955X

Artion Kashuri *is an associate professor at Department of Mathematical Engineering, Polytechnic University of Tirana. He received his Ph.D. in Mathematics from the University of Vlora, Albania. His research interest is in Mathematical Inequalities, Special Functions, Fractional Calculus, Quantum Calculus, Numerical Analysis and Applied Mathematics. He has more than 200 published papers in high reputation journals.*

https://orcid.org/0000-0003-0115-3079

An International Journal of Optimization and Control: Theories & Applications (http://www.ijocta.org)